Mathematical Reviews

OF MICHIGAN
JAN 1961
MATHEMATICS

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Vol. 21, No. 11 DECEMBER, 1960 Reviews 7135–7824

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(*Continued on cover 3*)

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4 (82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. In reference to reviews, the sign # refers to the number of the review.

Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, 190 Hope St., Providence 6, R.I.

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Editorial Office

MATHEMATICAL REVIEWS, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to individual members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

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Printed in Great Britain by William Clowes and Sons, Limited, London and Beccles

Mathematical Reviews

Vol. 21, No. 11 December, 1960 Reviews 7135–7824

BIBLIOGRAPHICAL NOTES

Il Periscopio. Segnalazioni e Recensioni Bibliografiche.
This publication of the Casa Editrice de Kaan, Via de Amicis 16, Milano, appears irregularly. Each issue concerns a certain field of science, engineering, medicine, art, architecture and so on. The first issue appeared in August, 1959.

Journal of Mathematical Physics.
Vol. 1, no. 1 of this journal is dated January–February, 1960. It is published bimonthly by the American Institute of Physics, 335 East 45th Street, New York 17, New York. The subscription price (U.S. and Canada) is $8.00 to members of the Institute, $9.00 to non-members.

Soviet Astronomy-AJ.
This translation of "Astronomičeskiĭ Žurnal" is published by the American Institute of Physics with the cooperation of the American Astronomical Society. The translation begins with the 1957 issues. One volume is published annually. Volume 3 will contain all of the scientific material published in the original Russian Volume 36 for 1959. Subscriptions should be sent to the American Institute of Physics, East 45th Street, New York 17, New York.

Z. Angew. Math. Mech. 39 (1959), no. 9/11 (pp. 337–460).
This issue is devoted to 72 abstracts of papers delivered at the "Wissenschaftliche Jahrestagung der Gesellschaft für Angewandte Mathematik und Mechanik" in Hannover, Germany on May 19–23, 1959.

Проблемы кибернетики. [Problemy kibernetiki].
This is being published as a non-periodical, under the editorship of A. A. Lyapunov. The first 3 issues appeared in 1958, 1959, 1960 respectively, and contain articles on general cybernetics, programming, computing machines, control systems, mathematical linguistics, and applications to biology.

HISTORY AND BIOGRAPHY

See also 7139.

7135:
Gillings, R. J. The Egyptian 2/3 table for fractions. The Rhind mathematical papyrus (B.M. 10057–8). Austral. J. Sci. **22** (1959), 247–250.

Ancient Egyptian arithmetic utilized only natural numbers, their reciprocals, the special fraction $\frac{2}{3}$, and combinations of these formed by addition. The Rhind Papyrus contains many examples in which two-thirds of a complicated combination of this sort is written down without intermediate steps being shown. The author of this paper infers that there existed a table for obtaining these results. He shows how such a table, of the form $(\frac{2}{3})a$, $a=1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \cdots, 100$, could have been constructed by the ancient Egyptians, and how such a table would suffice for the solution of all problems of this type in the Rhind Papyrus. *E. S. Kennedy* (Beirut)

7136:
★Chow, Hung-ching. The collected papers of Hung-ching Chow. Institute of Mathematics, Academia Sinica, Taipei, Taiwan, China, 1959. ix+181 pp. (1 plate)
Includes a photograph, preface by C. B. Ling, introduction by L. S. Bosanquet, and reproductions of the author's 22 published papers, all on summability.

GENERAL

See also 7700.

7137:
★Leja, Franciszek. Rachunek różniczkowy i całkowy: Ze wstępem do równań różniczkowych. [Differential and integral calculus: With an introduction to differential equations.] 5th revised ed. Biblioteka Matematyczna, Tom 2. Państwowe Wydawnictwo Naukowe, Warsaw, 1959. 466 pp. zł. 48.00.
This edition differs from the previous editions as follows: (1) The notion of upper bound, lower bound and a number of the theorems now appear earlier in the work; (2) certain of the proofs are now more fully treated; (3) the calculus is more extensively applied to problems of physics; (4) the number of examples has been increased. The second edition, published in 1949, received a Polish State award in 1953.

7138:
★Schaaf, William L. Basic concepts of elementary mathematics. John Wiley & Sons, Inc., New York-London, 1960. xvii+386 pp. $5.50.
A textbook directed toward improving the background of elementary and secondary school teachers, but also usable for general purposes, with many exercises and bibliographical references. Table of contents: Modern mathematics; elementary logic; geometry; the number

concept; extending the number system; numeration, exponents, and logarithms; measurement; mensuration; functions and graphs; interest and present value; probability and insurance.

7139:

★Ball, W. W. R.; Petersen, J.; Carslaw, H. S.; and Cajori, F. String figures and other monographs. Chelsea Publishing Co., New York, 1960. xxix+489 pp. (8 plates) $3.95.

Photographic reproduction of the following four books. Ball, *String figures* [3d ed., Heffer, Cambridge, 1928]; Petersen, *Methods and theories for the solution of problems of geometrical constructions applied to 410 problems* [Høst, Copenhagen, 1879]; Carslaw, *The elements of Non-Euclidean Plane Geometry and Trigonometry* [Longmans, Green & Co., London-New York, 1916]; Cajori, *A history of the logarithmic slide rule and allied instruments* [The Engineering News Publ. Co., New York, 1909].

7140:

★Reid, Constance. Introduction to higher mathematics for the general reader. Thomas Y. Crowell Co., New York, 1960. vii+184 pp. $3.50.

This book is intended as a popularized presentation of some advanced topics, covering number theory, Euclidean and non-Euclidean geometries, algebra, group theory, logic, calculus, topology, etc.

The selection of the topics and their sequence are intended to lead the prospective reader step-by-step into the more intricate phases of "Higher Mathematics", the attempt being made to present abstract mathematical concepts in a semi-conversational style, with some unrelated anecdotal passages. Much of the material is introduced with a promise to state the unfamiliar in terms familiar, but in the reviewer's opinion an uninitiated reader (and it is to such that the book is addressed) will only be confused by the large number of terms and statements which still require explanation.

A. Bakst (New York)

7141:

★Tietze, Heinrich. Gelöste und ungelöste mathematische Probleme aus alter und neuer Zeit. 2te durchgearbeitete Aufl. Verlag C. H. Beck, München, 1959. Bd. 1, xx+256 pp. (10 plates); Bd. 2, iv+298 pp. (8 plates). Zusammen geheftet: DM 28.00; Ganzleinen: DM 36.00.

A leisurely promenade through mathematics, written as a series of lectures for the layman, profusely illustrated and annotated. See MR **11**, 571, for a review of the first edition [Biederstein-Verlag, München, 1949].

7142:

★Félix, Lucienne. The modern aspect of mathematics. Translated by Julius H. Hlavaty and Fancille H. Hlavaty. Basic Books, Inc., New York, 1960. xiii+194 pp. $5.00.

Originally *L'aspect moderne des mathématiques*, Blanchard, Paris, 1957 [MR **20** #3062].

7143:

★Menninger, Karl. Mathematik und Kunst. Vandenhoeck & Ruprecht, Göttingen, 1959. 44 pp. (11 plates) Paperbound: DM 2.40.

A small nuggett, handsomely illustrated, discussing with many examples the several arts—poetry, music, architecture, etc.—in their mathematical aspects; mathematics being taken in its broadest sense, viz., of form (Gestalt).

LOGIC AND FOUNDATIONS

See also 7165, 7823.

7144:

★Anscombe, G. E. M. An introduction to Wittgenstein's Tractatus. Hutchinson University Library, London, 1959. 179 pp. 10s. 6d.

This book, written by an abundantly well-qualified philosopher, is an introduction, primarily for the aid of graduate students, to certain aspects of the only book [*Tractatus logico-philosophicus*, Routledge and Kegan Paul, Ltd., London, 1922, 1958] published by the late German-speaking philosopher and logician, Ludwig Wittgenstein [1889–1951].

Only rarely does the author attempt to paraphrase Wittgenstein. This is in keeping with her policy of treating many Wittgensteinian questions and aphorisms in a manner compatible with the psychological needs of the less mature student. Sometimes this results in her knowingly "integrating-out" rather than illuminating certain troublesome problems, e.g., the nature of elementary propositions and simple signs. In addition the reviewer believes that her attempts at paraphrasing seldom have either Wittgenstein's crispness or depth. However, she is quite clear in showing how certain of Wittgenstein's inferences were evolved from the earlier logical studies made by Frege and Russell. This is in keeping with the spirit of some of Cambridge's philosophical activities.

In spite of the negative ending to the *Tractatus* and the eventual rejection of much of its content by Wittgenstein himself, it still represents a monumental work of intense and penetrating thought. The author is to be commended for making such a fine introductory book available to students. The reviewer especially recommends using the book as a corequisite to a first reading of the *Tractatus*.

In closing, the reviewer notes that now only Wittgenstein's posthumously published *Foundations* [*Remarks on the foundations of mathematics*, Macmillan, New York, 1956; MR **19**, 1] is in need of "an introduction".

A. A. Mullin (Urbana, Ill.)

7145:

Geymonat, Ludovico. Logica matematica e algebra moderna. Confer. Sem. Mat. Univ. Bari **37**, 20 pp. (1958).

The author interprets classical logic as a Boolean algebra in the following way. Given a set A of axioms, statements p and q are equivalent if $p \leftrightarrow q$ is a formal consequence of A. Elements of the Boolean algebra are equivalence classes $|p|$ of statements p under this relation. The Boolean operations are defined as follows: $|p|+|q| = |p \vee q|$, $|p| \cdot |q| = |p \wedge q|$, $-|p| = |\sim p|$. With similar

correspondences in mind, the author discusses the algebraic structure of intuitionistic propositional calculi, and the cylindrical structure of predicate calculi. The introduction contains a simple exposition of Boolean algebras, and the style throughout is intuitive and oriented from a philosophical point of view.

E. J. Cogan (Bronxville, N.Y.)

7146:

da Costa, Newton Carneiro Affonso. Note on the concept of contradiction. Soc. Parana. Mat. Anuário (2) **1** (1958), 6–8. (Portuguese. English summary)

The author argues that, whereas an inconsistent mathematical theory has little mathematical value, it does serve a purpose from a philosophical point of view, for example, to be used comparatively in the metalogical evaluation of similar more useful theories. However "new logical rules", unspecified by the author, must be used in formalizing such theories.

E. J. Cogan (Bronxville, N.Y.)

7147:

da Costa, Newton Carneiro Affonso. Note on the logic of Brouwer-Heyting. Soc. Parana. Mat. Anuário (2) **1** (1958), 9–10. (Portuguese. English summary)

The author presents a definition of "meaningfulness" of a supposition in intuitionistic logic as follows: a supposition s is meaningful if and only if it can be determined of an arbitrary construction c whether or not it is appropriate for s. *E. J. Cogan* (Bronxville, N.Y.)

7148:

da Costa, Newton Carneiro Affonso. A question of philosophy of mathematics. Soc. Parana. Mat. Anuário (2) **1** (1958), 21–27. (Portuguese. English summary)

This paper presents an argument for the importance of pragmatic considerations in mathematics, as contrasted with considerations which are semantic and syntactic. The author cites intuitionism and the axiom of choice as examples of issues in mathematics in which decisions require fundamental pragmatic decisions, for example, as to the appropriateness of a theory to the problem at hand.

E. J. Cogan (Bronxville, N.Y.)

7149:

Ajdukiewicz, Kazimierz. Le problème du fondement des propositions analytiques. Studia Logica **8** (1958), 259–281. (Polish and Russian summaries)

This paper is concerned with the question: Are analytic propositions based entirely on linguistic conventions? Analytic propositions may be characterized, according to the author, either semantically or syntactically. Semantically, a proposition P is analytic for a language L if it is a "postulate" of L or a logical consequence of one or more postulates of L. A postulate P in turn is a terminological convention valid for L which requires that a certain term λ contained in P should designate the object satisfying the sentential function formed from P by replacing λ by a free variable. But terminological conventions are not alone sufficient to justify postulates—an existential assumption must also be made that there is an actual object designated by λ.

A proposition is analytic in the syntactic sense, according to the author, if it either is a logical truth or is formed from a logical truth with the help of a syntactical convention. In general, a syntactical convention is a declaration that if one accepts a proposition containing a term A, one is equally disposed to accept the one containing B in its place, and conversely. Here also, then, terminological conventions are not sufficient alone to establish analytic propositions—logical truths are also required. The paper concludes with a discussion of logical truths and existence.

R. M. Martin (Bonn)

7150:

Goodstein, R. L. On the formalisation of indirect discourse. J. Symbolic Logic **23** (1958), 417–419.

It is shown that Cohen's example [same J. **22** (1957), 225–232; MR **20** #4482] of a logical truth of indirect discourse is amenable to a straightforward formalisation and proof in a suitable variant of first order predicate calculus. Moreover, a simple consistency proof is given for the premisses involved in the proof.

E. W. Beth (Amsterdam)

7151:

Cohen, L. Jonathan. Professor Goodstein's formalisation of the policeman. J. Symbolic Logic **23** (1958), 420.

The question is raised whether Goodstein's formalisation [#7150 above] makes allowance for the substitutivity of the biconditional. If so {but, it would seem to the reviewer, it does not}, the resulting system would take no account of the referentially opaque quality of indirect discourse. It is argued, moreover, that difficulties would arise if, in addition to letters which function as sentential operators, we should admit letters functioning as genuine predicates. Presumably these difficulties could only be avoided by means of a two-sorted logic.

E. W. Beth (Amsterdam)

7152:

Suszko, Roman. Syntactic structure and semantical reference. I. Studia Logica **8** (1958), 213–247. (Polish and Russian summaries)

This is the first of two papers and is concerned mainly with syntactical preliminaries. The author presents a scheme of standard formalization, under which every symbolic language-system with suitable adjustment of calligraphy is presumed to fall. Ajdukiewicz' method of indices [Przegląd Filozoficzny **37** (1934), 219–235; Studia Philosophica **1** (1935), 1–27] is modified so as to take into account variables and quantifiers. The expressions of a language are represented by suitable geometric diagrams, and graphs are introduced to exhibit syntactic structure independent of any peculiarities of calligraphy. Relations of being-a-sub-language-of and of being-a-translation-of (with syntactic structure preserved) are defined.

R. M. Martin (Bonn)

7153:

Härtig, Klaus. Einbettung elementarer Theorien in endlich Axiomatisierbare. Z. Math. Logik Grundlagen Math. **4** (1958), 249–292.

Very detailed re-proof of the following result due to Kleene [Mem. Amer. Math. Soc., No. 10 (1952), 27–68; MR **14**, 439]: Let L be a first order language with a finite number of non-logical constants and let C be a recursively enumerable set of well formed formulas of L which

is closed under deduction. There exists an extension L' by a finite number of predicate constants of L and a finite set A of well formed formulas of L' such that a formula σ of L can be deduced from A (in L') if and only if $\sigma \in C$. *M. O. Rabin* (Jerusalem)

7154:

Lejewski, Czesław. On implicational definitions. Studia Logica 8 (1958), 189–211. (Polish and Russian summaries)

The author formulates a rule which he calls "implicational definition" which allows him to extend a fragmentary implicational calculus into a full implicational calculus. *N. S. Mendelsohn* (Winnipeg, Man.)

7155:

Quine, W. V. On cores and prime implicants of truth functions. Amer. Math. Monthly **66** (1959), 755–760.

Der Verf. gibt zunächst eine verbesserte Darstellung seines Algorithmus zur Gewinnung kürzester Normalformen von Formeln ϕ der Aussagenlogik. Jede Nf. ist eine Adjunktion (= Alternative) von Fundamentalformeln $P_1P_2\cdots P_n$ (in der jedes P_r eine Variable oder eine negierte Variable ist, keine Variable aber zweimal vorkommt). Für kürzeste Normalformen von ϕ kann man sich auf kürzeste Fundamentalformeln beschränken, die ϕ implizieren. Das sind die Primimplikanten. Um die Adjunktion aller Primimplikanten von ϕ zu erhalten, wird das Verfahren von Samson und Mills benutzt, anschliessend werden möglichst viele der Primimplikanten weggelassen. Diese Methode hat noch den Nachteil, daß die Anzahl der Primimplikanten für eine Formel mit n Variablen wesentlich größer als 2^n (die Zahl der Wahrheitswertbelegungen) sein kann. Als zweite Methode wird daher erörtert, von einer beliebigen Normalform auszugehen und von dieser alle überflüssigen Fundamentalformeln wegzulassen, ebenso von den Fundamentalformeln alle überflüssigen P_r. Eine auf diese Weise unverkürzbare Normalform heißt "irredundant". Die Fundamentalformeln von ϕ, die in jeder irredundanten Normalform von ϕ auftreten, bilden das "Herz" (core) von ϕ. Für die Zugehörigkeit einer Fundamentalformel zum Herzen gibt Verf. ein Kriterium an. Diese Methode führt jedoch nur in Spezialfällen zu kürzesten Normalformen. *P. Lorenzen* (Kiel)

7156:

★Surányi, János. Reduktionstheorie des Entscheidungsproblems im Prädikatenkalkül der ersten Stufe. Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1959. 216 pp.

A reduction type is a class T of formulas of the first-order predicate calculus such that there is an effective way of correlating with each formula φ of the first-order predicate calculus some formula φ' in T so that φ is satisfiable if and only if φ' is satisfiable. This book is principally a survey of the more interesting reduction types now known. The following is a sampling of the results.

Various elimination theorems, for function variables, propositional variables, truth values, equality signs, etc., are derived. The Skolem Normal Form is proved in a novel and simple way. The chief tool of the book is Theorem X, according to which the formulas $(\forall x_1x_2x_3M_1) \wedge (\forall y_1y_2\exists y_3M_2)$, where M_1 and M_2 are built up from individual variables, propositional connectives, arbitrarily many monadic predicate variables, and at most one binary predicate variable, form a reduction type. There is a sharpened form of the Skolem-Löwenheim Theorem, from which various reduction types can be obtained by arithmetic methods, e.g., the reduction type of formulas in prenex form with the prefix $\exists\forall\exists\forall^n$, and containing only one predicate variable, which is binary (Theorem XVIII). In Chapter VI, there is an assortment of reduction types in which the formulas are prenex and contain a single predicate variable, which is binary (e.g., with prefix $\forall^2\exists\forall^n$ or $\forall^3\exists^n$). Using the so-called Skolem-Herbrand Theorem, the author demonstrates Kalmar's theorem that to each formula $\mathfrak{U}$ there can be effectively associated a formula $\mathfrak{B}$ such that $\mathfrak{U}$ is not satisfiable if and only if $\mathfrak{B}$ is satisfiable in some finite domain. There is also a demonstration, based on a proof by Kalmar, of Church's Theorem on the undecidability of the first-order predicate calculus.

The author's discursive style makes the book quite readable, in spite of the length and complexity of some of the proofs. {The casual reader should be warned that the sign "$\vdash$" and the word "gleichwertig" are used in various different ways in the course of the book. Also, on page 34, it is falsely stated that if the structure S' is a homomorphic image of the structure S, then, if the formula $\mathfrak{U}$ holds in S, it also holds in S'. Counterexample: $6 \neq 0$ in the ring of integers, but $6 = 0$ in the homomorphic ring of integers modulo 6. However, as the author states, if $\mathfrak{U}$ does not contain the equality sign, it holds in S if and only if it holds in S'. This result is all that is needed in the sequel.} *E. Mendelson* (New York, N.Y.)

7157:

Lyndon, Roger C. Properties preserved under homomorphism. Pacific J. Math. **9** (1959), 143–154.

The author considers sentences with symbols for operations and relations (including identity) defined on a set, the variables whose range is the given set, and the usual logical symbols, i.e., the symbols for conjunction, disjunction, negation, and quantification. (The sentence contains no variable whose range consists of sets, relations, or functions.) A sentence will be called positive if it does not contain the symbol for negation. The author proves the following Theorem (*): A sentence of the predicate calculus is preserved under homomorphism if and only if it is equivalent to a positive sentence. (Theorem 5 is a stronger form of (*).) This solves a problem raised by E. Marczewski [Colloquium Math. **2** (1951), 220–228; MR **14**, 347]. The theorem (*) was stated also by J. Łoś [*Mathematical interpretation of formal systems*, North-Holland, Amsterdam, 1955; MR **17**, 700; p. 107] and A. I. Mal'cev [Dokl. Akad. Nauk SSSR **109** (1956), 264–266; MR **19**, 240]. *J. Jakubík* (Košice)

7158:

Thiele, Helmut. Theorie der endlichwertigen Łukasiewiczschen Prädikatenkalküle der ersten Stufe. Z. Math. Logik Grundlagen Math. **4** (1958), 108–142.

This paper, which is a report of the author's doctoral dissertation, concerns the semantic properties of the

Łukasiewicz n-valued first order predicate calculus. This is formulated in the first section with one designated value W and $n-1$ undesignated values $F_1, \cdots, F_{n-1}$, implication C^n defined so that $C^nF_mF_k = F_{k-m}$ where $F_j = W$ if $j \leq 0$, individual and predicate variables, and universal and existential quantifiers. Operations for interpreting the calculus in a domain J of individuals are provided, and the concepts of model, F-model, and universal truth of a formula relative to a model are defined. The basic relations among these ideas are investigated in section 2. Section 3 is devoted to reduction theorems and the representation problem, and section 4 to theorems which furnish necessary and sufficient semantic conditions for the axiomatizability of sets of formulas of the calculus. Section 5 investigates the properties of "Bolzano entailment" where "a set of formulas X entails a formula H in a domain of individuals J" means that any n-valued model of X in J is also an n-valued model of H.

E. J. Cogan (Bronxville, N.Y.)

7159:

★**Kreisel, G.; Lacombe, D.; and Shoenfield, J. R. Partial recursive functionals and effective operations.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 290–297. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

The paper is concerned with the comparison of two definitions of effective calculability of functionals. Let $\mathscr{A}$ be a class of recursive functions. Let $\{e\}$ denote the partial recursive function with Gödel number e. Following Myhill and Shepherdson [Z. Math. Logik Grundlagen Math. **1** (1955), 310–317; MR **17**, 1039] a functional F from $\mathscr{A}$ into natural numbers is called an effective operation if there exists a partial recursive function F^* such that $\{e\} \in \mathscr{A} \to F(\{e\}) = F^*(e)$. A functional $\mathfrak{F}$ from $\mathscr{A}$ into partial recursive functions is called an extensional effective operation if there exists a partial recursive function F_1 such that $\{e\} \in \mathscr{A} \to \mathfrak{F}(\{e\}) = \{F_1(e)\}$. A functional on $\mathscr{A}$ is called potentially partial recursive (p.p.r.) if it is extendable to a partial recursive functional in the sense of S. C. Kleene [*Introduction to metamathematics*, Van Nostrand, New York, 1952; MR **14**, 525]. Notions of a recursively dense base for a class $\mathscr{A}$ of recursive functions, and of recursive enumerability and recursiveness of $\mathscr{A}$, are introduced. Every recursively enumerable class $\mathscr{A}$ possesses a recursively dense base. Every p.p.r. functional is an effective operation. If $\mathscr{A}$ has a recursively dense base then every effective operation with domain $\mathscr{A}$ is p.p.r. A similar result holds for extensional effective operations. The non-extendability of the results achieved is shown by means of examples due to Myhill [no reference] and Friedberg [C. R. Acad. Sci. Paris **247** (1958), 852–854; MR **20** #6355]. The paper concludes with a necessary and sufficient condition for a class $\mathscr{A}$ to be recursive.

M. O. Rabin (Jerusalem)

SET THEORY

7160:

Bagemihl, Frederick. Some results connected with the continuum hypothesis. Z. Math. Logik Grundlagen Math. **5** (1959), 97–116.

Several propositions connected with the continuum hypothesis (H) are formulated and their mutual relationships and consequences examined. Here are some of these propositions. $\mathfrak{N}$ [resp. $\mathfrak{M}$]: The union of $<c$ linear sets of category I [measure 0] is of category I [measure 0]. $\mathfrak{L}$ [resp. $\mathfrak{S}$]: There exists a linear set of power c intersecting every linear perfect nowhere dense set [resp. every linear set of measure 0] in $<c$ points. $\mathfrak{N}^*$ [resp. $\mathfrak{M}^*$]: The linear continuum R_1 is not the union of $<c$ sets of category I [resp. of measure 0]. $\mathfrak{B}$ [resp. $\mathfrak{B}$]: The plane R_2 contains a set A intersected by every member of some set S of category II [set of exterior positive measure] of horizontal lines in $\leq \aleph_0$ points; moreover, every member of some non-enumerable set of vertical lines intersects $R_2 \setminus A$ in a linear set of category I [measure 0]. $\mathfrak{B}^*$ [resp. $\mathfrak{B}^*$] means $\mathfrak{B}$ [resp. $\mathfrak{B}$] restricted to the case that S is the set of all horizontal lines. $\mathfrak{N} \wedge \mathfrak{B} \Leftrightarrow$ H (Th. III.5). $\mathfrak{N}^* \wedge \mathfrak{B}^* \Leftrightarrow$ H (Th. III.5*). $\mathfrak{M} \wedge \mathfrak{B} \Leftrightarrow$ H (Th. III.6). One has also III.6*. The conjunction of $\mathfrak{N}$ and $\mathfrak{M}$ implies a permutation of R_1 the extension of which permutes the set of all linear sets of measure 0 as well as the set of all linear sets of category I (Th. XII.2). For other statements we refer to the paper itself.

D. Kurepa (Boulder, Colo.)

COMBINATORIAL ANALYSIS

See also 7215, 7551.

7161:

Dade, E. C.; and Goldberg, K. The construction of Hadamard matrices. Michigan Math. J. **6** (1959), 247–250.

The following theorem is proved: If there exists a transitive permutation group of degree $4n-1$ and odd order whose subgroups leaving one element fixed have three transitivity sets each, then a $(1, -1)$-matrix H of order $4n$ may be constructed such that $HH^T = 4nI$. The authors are aware that the groups in question are unlikely to exist when $4n-1$ is not a prime power.

H. K. Farahat (Sheffield)

7162:

Read, R. C. The enumeration of locally restricted graphs. I. J. London Math. Soc. **34** (1959), 417–436.

This paper is an important contribution to the art of enumeration. By a "locally restricted" graph is meant one in which there is a specified number of lines incident to the various points. The enumeration of this class of graphs does not appear solvable by the methods of Pólya [Acta Math. **68** (1937), 145–254].

After demonstrating a combinatorial lemma, the author proves the "Superposition Theorem", which gives a formula for the number of distinct "superposed graphs" that can be obtained by superposing the lines of a collection of k graphs each having n points. This number is given in terms of the cycle index of the automorphism groups of these graphs. The applications include the number of cubic graphs with colored lines, bicolored graphs with given valencies (degrees), graphs and cubic graphs with labeled points and lines, directed graphs with given valencies, and regular graphs with colored labeled lines. It must be noted that each of these results is obtained only for general graphs, that is, graphs in which both lines in parallel and loops (lines joining a point with

itself) are permitted. The last section notes a formula which counts the number of connected graphs of a given kind in terms of the number of all graphs of that kind and conversely. This result is already contained in a paper of the reviewer [Trans. Amer. Math. Soc. **78** (1955), 445–463; MR **16**, 844].

{There is a remarkable paper by J. H. Redfield [Amer. J. Math. **49** (1927), 433–455], which appears to either contain or anticipate virtually all of the enumeration results for graphs which have been discovered and developed during the last 30 years. In particular, Redfield's pioneering paper contains the exact statement of the Superposition Theorem and a sufficient number of illustrations to indicate a thorough knowledge of Pólya's enumeration method.} *F. Harary* (Ann Arbor, Mich.)

ORDER, LATTICES

See also 7167, 7175, 7176, 7222, 7824.

7163:

Crawley, Peter; and Dean, Richard A. Free lattices with infinite operations. Trans. Amer. Math. Soc. **92** (1959), 35–47.

A lattice L is called weakly $\aleph_\alpha$-complete if every subset of L of cardinality less than $\aleph_\alpha$ has a least upper bound and a greatest lower bound in L. A sublattice M of L is an α-sublattice if meets and joins in L of subsets of M of cardinality less than $\aleph_\alpha$ are also contained in M. For a given partially ordered set P, the free weakly $\aleph_\alpha$-complete lattice $F_\alpha(P)$ generated by P is defined in a natural way. $F_\alpha(P)$ is said to be freely α-generated by P. A construction of $F_\alpha(P)$ is given. Theorems: If P is a partially ordered set of cardinality at most $\aleph_\alpha$, then $F_\alpha(P)$ is isomorphic to an α-sublattice of the lattice $F_\alpha(3)$ freely α-generated by three unordered elements. (The special case of this theorem, when P is totally unordered, extends the result of Whitman [Ann. of Math. (2) **43** (1942), 104–115; MR **3**, 261] which concerns the case $\alpha=0$.) Every lattice of cardinality at most $\aleph_\alpha$ is a sublattice of a lattice α-generated by three elements. Under the Generalized Continuum Hypothesis, $F_\alpha(3)$ is an $\aleph_\alpha$-universal partially ordered set (i.e., every partially ordered set of cardinality at most $\aleph_\alpha$ is isomorphic to a subpartially ordered set of $F_\alpha(3)$) if $\aleph_\alpha$ is regular, and contains a sublattice which is an $\aleph_\alpha$-universal partially ordered set if $\aleph_\alpha$ is singular. There exist no $\aleph_0$-universal lattices. (A nonconstructive proof of the existence of $\aleph_\alpha$-universal lattices for all $\alpha>0$, and that of the $\aleph_\alpha$-universal partially ordered sets for all ordinals α, is given by Bjarni Jónsson [Math. Scand. **4** (1956), 193–208; MR **20** #3091].) *M. Kolibiar* (Bratislava)

7164:

Grätzer, Gy.; and Schmidt, E. T. On a theorem of Gábor Szász. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 255–258. (Hungarian)

A theorem of G. Szász [Acta Sci. Math. Szeged. **19** (1958), 224–228; MR **20** #6373] is proved in the following equivalent form: a finite complemented modular lattice L with 0 and 1 has a five-element non-distributive sublattice containing 0 and 1 if and only if every element with unique complement is of even height.

L. Fuchs (Budapest)

7165:

Nerode, A. Some Stone spaces and recursion theory. Duke Math. J. **26** (1959), 397–406.

With a distributive lattice A (with 0 and 1), associate the topological space A^* consisting of proper prime dual ideals of A. A^* is a Stone space [M. H. Stone, Časopis Pěst. Mat. Fys. **67** (1937), 1–25], and there is a natural embedding of A into A^{**}. If A and B are distributive lattices for which there exists a homomorphism $f:A\to B$, then there is a (strongly continuous) dual $f^*:B^*\to A^*$; each of f and f^* uniquely determines the other. In this setting the author obtains topological representation spaces of distributive lattices via theorems such as those due to Areškin [Mat. Sb. (N.S.) **33 (75)** (1953), 133–156; MR **15**, 334]. The central device which is used is the dual to the embedding of a distributive lattice into a Boolean algebra [W. Peremans, Nederl. Akad. Wetensch. Proc. Ser. A **60** (1957), 73–81; MR **18**, 868]. Finally, the author identifies the partial recursive functionals of recursion theory [A. V. Kuznecov and B. A. Trahtenbrot, Dokl. Akad. Nauk SSSR **105** (1955), 897–900; MR **17**, 1039] as continuous mappings between representation spaces and accordingly as duals to the elements of a class of homomorphisms between distributive lattices.

R. M. Baer (Berkeley, Calif.)

7166:

L'Abbé, Maurice. Structures algébriques suggérées par la logique mathématique. Bull. Soc. Math. France **86** (1958), 299–314.

This is an expository article. The algebraic structures discussed are: Boolean algebras, Post algebras, Boolean algebras with operators, relation algebras, projective algebras, closure algebras, cylindric algebras and polyadic algebras. {One error should be pointed out: In discussing the representation problem for relation algebras the author quotes Lyndon's paper [Ann. of Math. (2) **51** (1950), 707–729; MR **12**, 237] where it is asserted that the class of all representable relation algebras is not axiomatizable. That paper was later found to contain a mistake. In fact, Tarski showed that this class is axiomatizable, and such an axiom system was explicitly given by Lyndon [Ann. of Math. (2) **63** (1956), 294–307; MR **18**, 106].}

B. Jónsson (Minneapolis, Minn.)

GENERAL MATHEMATICAL SYSTEMS

See also 7264, 7391.

7167:

Hanf, William. On some fundamental problems concerning isomorphism of Boolean algebras. Math. Scand. **5** (1957), 205–217.

This paper and the three immediately following by Tarski and Jónsson [#7168, #7169, #7170] are closely related in content. They deal with fundamental problems concerning algebraic systems which were unsolved before or solved only in special cases. The reviewer feels that it is necessary to point out that the results of these papers are really important. The problems now solved were previously stated independently by several authors.

In what follows the direct product of n algebraic systems A will be denoted by A^n.

The present paper deals with Boolean algebras. Theorem 1: There exist denumerable Boolean algebras A and B such that $A \cong A \times B^2$ but A non$\cong A \times B$. More generally, given a positive integer n, there exist denumerable Boolean algebras A and B such that, for each integer m, $A^m \cong A^m \times B^k$ just in the case k is a multiple of n. A consequence is Theorem 2: There exist denumerable Boolean algebras A and C such that $A^2 \cong C^2$ but A non$\cong C$. More generally, given a positive integer n, there exist denumerable Boolean algebras A and C such that for every positive integer k, $A^k \cong C^k$ in the case k is a multiple of n, and A^k non$\cong C^k$ otherwise.

Then a theorem of R. L. Vaught is given which contains in particular the following result: If A is a denumerable Boolean algebra and B, C are finite Boolean algebras, then $A \cong A \times B \times C$ implies $A \cong A \times B \cong A \times C$. The next results concern general Boolean algebras. Theorem 4: Let T be a two-element Boolean algebra. There exists a Boolean algebra A such that $A \cong A \times T^n$ but A non$\cong A \times T^k$ for $k = 1, 2, \cdots, n-1$. Theorem 5: There exists a Boolean algebra A such that $A \cong A^n$ but A non$\cong A^k$ for $k = 2, 3, \cdots, n-1$.

Some interesting applications of the foregoing results are given. One of them (due to C. C. Chang) is as follows: There exist partially ordered systems A, B and a finite partially ordered system C such that $A \times C \cong B \times C$ but A non$\cong B$. *Št. Schwarz* (Bratislava)

7168:

Tarski, Alfred. Remarks on direct products of commutative semigroups. Math. Scand. **5** (1957), 218–223.

It is shown that the Theorems 1, 2, 4 and 5 of the foregoing paper of Hanf remain valid if the words "there exist Boolean algebras" are replaced by "there exist commutative cancellation semigroups with a zero" and if in addition the Boolean algebra T in Theorem 4 is replaced by the additive semigroup N of non-negative integers.

It is pointed out that the results of this and the foregoing paper solve analogous questions concerning the class of partially ordered commutative semigroups.

Some problems which still remain open are mentioned. *Št. Schwarz* (Bratislava)

7169:

Jónsson, Bjarni. On isomorphism types of groups and other algebraic systems. Math. Scand. **5** (1957), 224–229.

Hanf's results [#7167 above] are used to obtain analogous results for groups. It is proved that Theorems 1, 2, 4 and 5 of Hanf's paper remain valid if the words "Boolean algebras" are replaced by the words "centerless groups" and if, in addition, the algebra T in Theorem 4 is replaced by an indecomposable centerless group.

A general theorem is proved which enables most of Hanf's results to be carried over to any class of algebras with a zero element and with finitary operations, which contains a finite or denumerable indecomposable centerless algebra, and which is closed under the operation of taking subdirect products. (Among such classes are the class of all groups, the class of all commutative partially ordered semigroups and the class of all rings, as well as various subclasses of these classes.) *Št. Schwarz* (Bratislava)

7170:

Jónsson, Bjarni. On direct decompositions of torsionfree abelian groups. Math. Scand. **5** (1957), 230–235.

An example of a torsionfree abelian group G of finite rank having two decompositions $G = A \times B = C \times D$ such that $A \cong B$, $C \cong D$, but A and B are not isomorphic to C and D, is explicitly constructed.

The problem itself is in the same direction as the problems in #7169 above, but, since it deals with abelian groups, the existence of such groups cannot be derived by the same means.

On an earlier example due to the author it is shown that the direct product of two torsionfree abelian groups of finite rank may have a factor of rank one although neither group has such a factor.

Another example of a group G is given such that $G = A \times B = C \times D$, where $A \cong C$ but B and D are not isomorphic. *Št. Schwarz* (Bratislava)

7171:

Feferman, S.; and Vaught, R. L. The first order properties of products of algebraic systems. Fund. Math. **47** (1959), 57–103.

A very comprehensive theory is developed, with many applications, of general products of algebraic systems, that is, of uniform methods for combining a set of algebras A_i, where i ranges over an index set I, to form a new algebra A. The majority of familiar 'products' and 'sums' falls within the scope of this theory, although the authors point out that it does not include the free product of groups. The primary interest is with the relation of the elementary (first order) theory of the product A to the theories of the components A_i. The authors credit the germ of their method to Mostowski [J. Symbolic Logic **17** (1952), 1–31; MR **13**, 897], whose results they extend. In addition to many results new with the authors, both jointly and separately, the treatment unifies and extends work of Beth, Fraïssé, Tarski, Scott and others. {For further references see the bibliography of the paper under review; reference should be made also to the very recent and not fully published work of the 'Berkeley school' on reduced products, for example to Keisler [Notices Amer. Math. Soc. **7** (1960), 70–71].} To the reviewer, the fact that the authors provide the 'right way' of looking at these questions seems of no less value than the wealth of interesting results obtained; we attempt to illustrate each of these aspects in turn.

For simplicity we speak not of 'algebras' with operations, but of relational systems. The domain of a product A consists, typically, of the Cartesian product of the domains of the components A_i, i in I. In the full direct product of (similar) systems, an n-ary primitive relation θ is defined to hold for elements $f_1, \cdots, f_n$ in A when

$$\{i : \theta(f_1(i), \cdots, f_n(i)) \text{ holds in } A_i\} = I.$$

The essential generalization of this is to define new relations Q from given relations $\theta_1, \cdots, \theta_m$, which are to hold for $f_1, \cdots, f_n$ in A when the subsets

$$K_j = K(\theta_j; f_1, \cdots, f_n) = \{i : \theta_j(f_1(i), \cdots, f_n(i)) \text{ holds in } A_i\}$$

of I satisfy prescribed conditions $\Phi(K_1, \cdots, K_m)$. At simplest, Φ is some condition in the Boolean algebra S of all subsets of I, but for the more interesting constructions S is taken as a model for a language L richer than

that of Boolean algebra, with Φ a relation expressible in L. For example, if L is equipped with an additional primitive $F(X)$ expressing that X is finite, the 'almost everywhere' product of Chang and Morel is obtained by introducing into A, for each primitive θ of the A_t, a relation θ' defined by the condition $F(I-K(\theta; f_1, \cdots, f_n))$. If, more generally, $F(X)$ characterizes the elements X of an arbitrary ideal in the Boolean algebra S, one obtains analogously the 'reduced products' of Łoś, Kochen, and the Berkeley school. In a different direction, 'ordinal products' are obtained by equipping S with a relation that reflects an order relation on I. Again, reading a predicate $\theta_0(x)$ of the A_t to say that x is a 'neutral element', restriction of the direct product A to the subset of those f such that $I-K(\theta_0; f)$ is finite yields the weak direct product; relativization by the condition that $I-K(\theta_0; f)$ has exactly one element leads to the 'cardinal sum'. Generalized powers $A=B^S$ are obtained by taking all A_t isomorphic.

A sample of the more easily quoted results follows. A basic result shows that any relation R on A, definable by a first order formula built up from relations Q of the sort described above, is itself such a relation Q; the hard part of the proof involves an 'elimination of quantifiers'. An important consequence is that if each A_t is replaced by an 'elementarily equivalent' A_t' (that is, satisfying the same first order sentences), then the products A and A' are elementarily equivalent; with a minor qualification, the corresponding result holds for 'elementary extensions'. Again, if B and B', and also S and S', are elementarily equivalent, then B^S and $B'^{S'}$ are elementarily equivalent. If a first order sentence holds in each direct product $A_1 \times \cdots \times A_n$, $n=1, 2, \cdots$, then it holds in $\prod A_n$; this answers a question raised by Łoś. If I is finite (and S has only a finite set of primitives), decidability for A reduces to that for the A_t. Decidability for B^S reduces to that for B and S; this is used to recover Mostowski's theorem that decidability for a direct power of B reduces to that for B. Decidability for the theory of the class of all A constructible from A_t from a given class $\mathscr{A}$ with S from a given class $\mathscr{S}$, reduces to decidability for the theories of the classes $\mathscr{A}$ and $\mathscr{S}$. Applications to various special products are given, to abelian groups, to certain parts of number theory, and to the arithmetic of ordinals.

R. C. Lyndon (Princeton, N.J.)

7172:

Foster, Alfred L. An existence theorem for functionally complete universal algebras. Math. Z. 71 (1959), 69–82.

An algebra $(P, O_1, O_2, \cdots)$ is said to be primary provided every mapping of P^n into P, $n=1, 2, \cdots$, can be obtained by composition from the operations O_t and the identity maps $(x_1, x_2, \cdots, x_n) \to x_k$. The principal result states that for any finite binary algebra $(P, \times)$ with null and identity $(0 \times a = a \times 0 = 0,\ 1 \times a = a \times 1 = a,\ 0 \neq 1)$ there exists a cyclic permutation $a \to a^\frown$ of P such that $(P, \times, ^\frown)$ is primary. With just one exception the permutation $^\frown$ can be so chosen that $0^\frown = 1$. The exceptional algebra is of order 4, $P=\{0, 1, \alpha, \beta\}$, with $\alpha \times \alpha = \alpha$, $\beta \times \beta = \beta$, $\alpha \times \beta = \beta \times \alpha = 0$.

B. Jónsson (Minneapolis, Minn.)

7173:

Słomiński, J. The theory of abstract algebras with infinitary operations. Rozprawy Mat. 18, 67 pp. (1959).

"No systematic presentation of the foundations of the theory of algebras with infinitary operations has been published so far. The purpose of this paper is to give such presentation and also an elaboration of the theory of equationally definable classes of algebras with infinitary operations." (p. 3.) The paper is a doctoral dissertation presented to the University of Toruń in 1957 and consists of an introduction (pp. 3–5), Ch. I: Operations (6–9), Ch. II: Abstract algebras (10–20), Ch. III: Equationally definable classes of algebras (21–45), Appendix to Ch. III: Functionally free algebras (46–48), Ch. IV: Gödel's theorem for O-systems (49–65).

For any ordinal α and any set A the mappings of A^α into A are called α-ary operations on A; these operations are finitary ($\alpha < \omega$) or infinitary ($\alpha \geq \omega$); in particular, 0-ary operations on A are constant members of A. Let β be an ordinal and $\Delta=(\alpha_\xi)_{\xi<\beta}$ a β-sequence of ordinals; let F_ξ be an operation of type α_ξ on A; the sequence $\mathbf{A}=(A, F_0, \cdots, F_\xi, \cdots)_{\xi<\beta}$ is called an algebra of type Δ; the cardinal number kA of A is called the power of the algebra; $\sup k\alpha_\xi = \rho$ is the rank of Δ; dimension of Δ is the minimal initial ordinal $\gamma > \alpha_\xi$ non-cofinal with any number $\leq \alpha_\xi$. For any $A_0 \subset A$, one defines by induction the Borel classes of A in the algebra $\mathbf{A}$ as the sets

$$A_\sigma = \bigcup_{\tau<\sigma} A_\tau \cup \bigcup_{\xi<\beta} F_\xi\Big(\bigcup_{\tau<\sigma} A_\tau\Big)$$

for every $\sigma < \gamma$. The algebra $\{A_0\}$ generated by A_0 equals $(\bigcup_{\sigma<\gamma} A_\sigma, F_0, \cdots, F_\xi, \cdots)_{\xi<\beta}$, and $k\{A_0\} = (kA_0 \cdot k_\beta \cdot k_\rho)^{k_0}$. An algebra $\mathbf{W}$ of type Δ is absolutely free of type Δ if: (1) $\mathbf{W}$ is generated by some $X_0 \subset W$ and the operations F_ξ ($\xi < \beta$) do not take on any value in X_0; (2) for $\eta, \lambda < \beta$ and $(w_\tau)_{\tau<\alpha_\eta}$, $(w_\tau')_{\tau<\alpha_\lambda}$ from W, if $F_\eta w = F_\eta w'$, then $F_\eta = F_\lambda$ and $w_\tau = w_\tau'$ for $\tau < \alpha_\eta = \alpha_\lambda$. There exist absolutely free algebras of type Δ freely generated by sets of any power. Elements w of an absolutely free algebra $\mathbf{W}$ of type Δ generated by X_0 are called Δ-W-terms; ordered pairs of terms are called equations; w is said to be of degree σ provided the X_0-Borel class of w equals σ; members of X_0 are called variables. Validity of an equation in an algebra is defined using homomorphism: A Δ-W-equation (a, b) is valid in an algebra $\mathbf{D}$ if for every $\varphi: X_0 \to D$ one has $h_\varphi a = h_\varphi b$, where h_φ is a homomorphism from $\mathbf{W}$ in D extending the mapping φ.

Let $E_\Delta^{\mathbf{W}}$ be the set of all Δ-W-equations; in particular, the author considers the case of an algebra $\mathbf{W}^*$ absolutely free of type Δ freely generated by a set X^* of cardinality $k\rho$; the Δ-$\mathbf{W}^*$-equations are called equations, and one puts $E_\Delta^{\mathbf{W}^*} = E_\Delta$, $E_\Delta^{\mathbf{W}}(\mathbf{D}) = E_\Delta(\mathbf{D})$, $\Delta^{\mathbf{W}^*}(E) = \Delta(E)$, where $E_\Delta^{\mathbf{W}}(\mathbf{D})$ denotes the set of all Δ-W-equations valid in $\mathbf{D}$; for $E \subset E_\Delta^{\mathbf{W}}$, $\Delta^{\mathbf{W}}(E)$ is the class of all algebras of type Δ in which E is valid. For a class $\mathfrak{A}_0$ of algebras, one denotes by $E_\Delta^{\mathbf{W}}(\mathfrak{A}_0)$ the set of all equations valid in $\mathfrak{A}_0$. For every algebra $\mathbf{D}$ of type Δ and every homomorphism h of D in any algebra, one has $E_\Delta(D) \subset E_\Delta(hD)$ (§ 5.1). An equationally definable class of algebras is any class $\mathfrak{A}_0$ of algebras of type Δ such that $\mathfrak{A}_0 = \Delta(E)$ for some $E \subset E_\Delta$. A class $\mathfrak{A}_0$ of algebras of type Δ satisfying $E_\Delta(\mathbf{D}) = E_\Delta(\mathfrak{A}_0)$ for some algebra $\mathbf{D}$ is an equationally definable class of algebras if and only if $H(S(P(\mathbf{D}))) \subset \mathfrak{A}_0$. $S\mathfrak{A}_0$, $H\mathfrak{A}_0$, $P\mathfrak{A}_0$ denote respectively: (1) the class of all subalgebras in $\mathfrak{A}_0$; (2) the class of all homomorphic images of algebras in $\mathfrak{A}_0$; (3) the class of algebras $\prod_{t \in T} \mathbf{D}_t$ where $\mathbf{D}_t \in \mathfrak{A}_0$ (T any set). In Ch. IV one considers an elementary deductive theory in connection with the class of similar

algebras of type Δ and in which the formulae—called O-formulae—do not contain quantifiers. In connection with O-formulae one considers an algebra $\mathbf{O}=(O, \rightarrow, ')$ freely generated by sets of all equations, and the author proves the analogous of Gödel's theorem: Every consistent O-system has a model. The paper is connected with works of G. Birkhoff [Proc. Cambridge Philos. Soc. **31** (1935), 433–454—here infinitary algebras originated], Tarski [Fund. Math. **33** (1945), 51–65; MR **7**, 360], Łoś [Studia Logica **2** (1955), 151–212; MR **18**, 785] (this in particular in connection with the proof of the main theorem in Ch. IV). *Đ. Kurepa* (Boulder, Colo.)

7174:

Yanov, Yu. I.; and Mučnik, A. A. Existence of k-valued closed classes without a finite basis. Dokl. Akad. Nauk SSSR **127** (1959), 44–46. (Russian)

Let P_k be the class of functions (of an arbitrary finite number of arguments) whose arguments and values are in the set $\{0, 1, \cdots, k-1\}$. A subset S of P_k is closed if every composition of functions in S is in S. A subset B of S is a basis if every function in S is a composition of functions in B. The authors exhibit, for each $k \geq 3$, a subset of P_k having no basis, and a subset of P_k having an infinite basis. The final result states that, for every $k \geq 3$, the class of closed subsets of P_k has the power of the continuum. *E. J. Cogan* (Bronxville, N.Y.)

7175:

Jónsson, Bjarni. Representation of modular lattices and of relation algebras. Trans. Amer. Math. Soc. **92** (1959), 449–464.

An algebraic system A with operations $x \wedge y$, $x;y$, x^{-1} and (a constant) I is 'representable' if it is isomorphic to some algebra of binary relations where the operations are, respectively, intersection, relative product, converse, and identity. Theorem 1, the central result, gives an infinite set Γ of conditions in the form of universal sentences which is necessary and sufficient for a system A to be representable. The conditions Γ constitute a great simplification, possible in the present context, of conditions given by the reviewer [Ann. of Math. (2) **51** (1950), 707–729; **63** (1956), 294–307; MR **12**, 237; **18**, 106] for a 'relation algebra' to be representable (see below), and the proof parallels that of the reviewer, using results of Henkin and Tarski. Theorem 2 provides analogous conditions for a (modular) lattice to be isomorphic to a lattice of commuting equivalence relations. The proof rests upon the observation that a lattice of commuting equivalence relations constitutes a system A that is 'symmetric': $x=x^{-1}$, except that it may lack a least element I.

A 'relation algebra' was defined by Tarski [J. Symb. Logic **6** (1941), 73–89; MR **3**, 130] to be a system A' with an additional operation x', to be interpreted as complementation under representation, and subject to a certain finite set of axioms. Theorem 3 provides conditions Γ' for A' to be 'weakly representable', that is, for the curtailed system A to have a representation carrying 0 into the empty relation.

Theorem 4 asserts the existence of symmetric relation algebras that are also integral: $x;y=0$ implies $x=0$ or $y=0$, which are not weakly representable. This is an improvement over a non-representable relation algebra constructed by the reviewer, which was neither symmetric nor integral. The proof rests upon constructing, from a projective plane P, an algebra A_0 that is a complete atomistic Boolean algebra with atoms I together with p^* for each point p of P, and a multiplication essentially determined by the condition that $p^* \subset q^*;r^*$ just in case p, q, and r are distinct and collinear; it is then shown that A_0 can be embedded in a symmetric integral relation algebra A. {The author has noted that if one modifies his construction by further taking $p^* \subset p^*;p^*$, then A_0 is already such an algebra A.} It is shown that the first of the conditions Γ' implies Desargues' Theorem for P, whence if P is not Desarguesian, the algebra A is not weakly representable.

The author concludes by asking if the conditions Γ, or Γ', can be replaced by a finite set; whether every modular lattice is isomorphic to a lattice of commuting equivalence relations; and whether weak representability may not in fact imply representability.

R. C. Lyndon (Princeton, N.J.)

7176:

Wiegandt, Richard. On the general theory of Möbius inversion formula and Möbius product. Acta Sci. Math. Szeged **20** (1959), 164–180.

Let S be a partially ordered set (with order denoted by $<$) having the property that for every $a \in S$, there are only a finite number of elements d of S satisfying $d \leq a$. First, consider functions from S to a module. A function $f(a)$ determines a sum function (1) $F(a)=\sum_{d \leq a} f(d)$ and, conversely, a function $F(a)$ determines an inversion function $f(a)$ satisfying (1) and $f(a)$ is given by (2) $f(a)=\sum_{d \leq a} c_{ad} F(d)$, where c_{ad} are integers. Let $\delta(a)$ be the Dirac function, $\delta(a)$ is 1 or 0 according as a is or is not a minimal element of S, and let $\mu(a)$ be the Möbius function, defined as the inversion function of $\delta(a)$. The first part of the note concerns the existence of functions $\rho(a, b)$ from $S \times S$ to S such that $c_{ab}=\mu(\rho(a, b))$. Next, let $\mathfrak{F}$ be the set of functions from S to a field, $\alpha(a, b)$ a function from $S \times S$ to S, and define the α-product of $f, g \in \mathfrak{F}$ by $(f \cdot g)_\alpha(a)=\sum_{d \leq a} f(d) g(\alpha(a, d))$. Necessary and sufficient conditions on α are determined in order that $\mathfrak{F}$ becomes a commutative semi-group under α-multiplication; in which case, $\delta(a)$ is the identity element. Applications of the above are made to obtain results of S. Delsarte dealing with certain functions of Abelian groups [Ann. of Math. (2) **49** (1948), 600–609; MR **10**, 9], and to the case that S is the collection of finite subsets (of a denumerable set) partially ordered by inclusion. *P. Hartman* (Baltimore, Md.)

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See also 7214, 7222, 7226, 7805.

7177:

Schulz, Werner. Einige Bemerkungen über Differenzenschemata mod n. Math. Nachr. **19** (1958), 129–135.

The author considers sequences of integers a_j which are periodic modulo some integer n (i.e., $a_{j \pm n}=a_j$). Theorem 1: Suppose that $n=p^\alpha$, where p is prime. Then $\Delta^q a_j \equiv 0 \ (p^\alpha)$, where $q=\alpha p^\alpha$. If $\alpha=1$ this result may become false if $q-1$ is substituted for q. Theorem 2: Suppose that $n=p$

and that all the a_j are 0, $+1$, or -1, with $a_0=0$, $a_1=1$. Then $\Delta^q a_j \equiv 0\ (p)$ implies that $2q \geq p+1$. It implies the stronger estimate $2q>p+1$ except when $a_j=\left(\frac{j}{p}\right)$ is the symbol of quadratic residuacity.

The proof of Theorem 1 is a simple induction. That of Theorem 2 follows from the consideration of the polynomials mod p whose roots are respectively the indices j such that $a_j=1$, $a_j=-1$.

J. W. S. Cassels (Cambridge, England)

7178:

Carlitz, L. A note on integral-valued polynomials. Nederl. Akad. Wetensch. Proc. Ser. A **62**=Indag. Math. **21** (1959), 294–299.

Let J be the class of all integral-valued polynomials; K_r, K, M_r, M are sub-classes of J. K_r consists of those $f \in J$ for which the νth derivative $f^{(\nu)}$ satisfies $f^{(\nu)} \in J$ $(\nu=1, \cdots, r)$; K is the intersection of $K_1, K_2, K_3, \cdots$. We put $M_0=J$. $M_1, M_2, \cdots$ are defined by induction: if $r \geq 0$, then M_{r+1} is the class of all $f \in J$ which are such that the polynomial $(f(x+m)-f(x))/m$ belongs to M_r for all integers $m \neq 0$. M is the intersection of $M_0, M_1, \cdots$. The author characterizes the various classes by divisibility properties of the coefficients $b_0, b_1, b_2, \cdots$ in the representation

$$f(x) = b_0+b_1\binom{x}{1}+b_2\binom{x}{2}+\cdots$$

($b_n=0$ from a certain n onward). For example, we have $f \in M_r$ if and only if b_n is a multiple of $L_n^{(r)}$, i.e., the least common multiple of all products $s_1 \cdots s_r$, where $s_1, \cdots, s_r$ are positive numbers restricted by $s_1+\cdots+s_r \leq n$. This generalizes the characterization of M_1 given by the reviewer [same Proc. **58** (1955), 363–367; MR **17**, 128], as well as the one given by E. G. Straus [Proc. Amer. Math. Soc. **2** (1951), 24–27; MR **12**, 700, 1003]. Straus' result refers to K, but the present author proves that $K=M$. He shows, however, that $M_1 \neq K_1$, although $M_r \subset K_r$ for all r.

{The reviewer could not follow the "only if" part of the proof of theorem 4, but he remarks that his own proof of the "only if" part in the case of M_1 can be immediately generalized to M_r. Misprint: in theorem 2, a_{n+k} should be a_{s+k}.}

N. G. de Bruijn (Amsterdam)

7179:

Ellis, David. Some consequences of Wilson's theorem. Univ. Nac. Tucumán. Rev. Ser. A **12**, 27–29 (1959).

Let $\pi(k, j)=[(j-1)!(k-j)!+(-1)^{j+1}]/k$. For all $k \geq 2$, $1 \leq j \leq k$, $\pi(k, j)$ is an integer if and only if k is a prime. If k is an odd prime, it divides $1+\sum_{r=0}^{k-1} r!(k-1-r)!$. The author also gives a complicated "closed" expression for the number of primes up to n.

N. J. Fine (Princeton, N.J.)

7180:

Ward, Morgan. Tests for primality based on Sylvester's cyclotomic numbers. Pacific J. Math. **9** (1959), 1269–1272.

Let n be an integer and let

$$F_n(x) = \prod_{\delta \mid n} (x^{n/\delta}-1)^{\mu(\delta)}$$

be the irreducible cyclotomic polynomial of degree $\phi(n)$. Let α and β be the roots of $x^2-Px+Q=0$ where P and Q are fixed coprime integers. Then Sylvester's cyclotomic number Q_n is defined for $n>1$ by

$$Q_n = Q_n(\alpha, \beta) = \beta^{\phi(n)} F_n(\alpha/\beta).$$

The author sets $Q_1=1$ so that

$$\prod_{\delta \mid n} Q_\delta = \frac{\alpha^n-\beta^n}{\alpha-\beta} = U_n$$

is Lucas' function. The author proves that if m is any odd divisor of Q_n with n and m coprime then every divisor of m, except 1, has the same rank of opposition n in the sequence $U_1, U_2, \cdots$. That is, m divides U_n but no divisor of m divides an earlier number U_k $(k<n)$. As a corollary the following sufficient test for primality is obtained. Let $m>1$ be odd and let m divide Q_n. Suppose that $n>m^{1/2}$ and n and m are coprime. Then m is a prime except in two trivial cases: $m=(n-1)^2$, $n-1$ a prime >3, or $m=(n-1)(n+1)$ a product of two twin primes.

The case $n=2^\lambda$ and $m=2^\lambda \pm 1$ give the classical Lucas tests. The practical application of the test to other cases is not considered. Examples are given with $n=6$ to show that the trivial exceptions mentioned above actually occur.

D. H. Lehmer (Berkeley, Calif.)

7181:

Durst, L. K. Exceptional real Lehmer sequences. Pacific J. Math. **9** (1959), 437–441.

The author deals with certain sequences $\{P_n\}$ of rational integers, known as (real) Lehmer sequences. Correcting results of Ward [Ann. of Math. (2) **62** (1955), 230–336; MR **17**, 127] he determines the exceptional cases in which there is no prime p with $p \mid P_n$, $p \nmid P_m$ for $0<m<n$.

C. G. Lekkerkerker (Amsterdam)

7182:

Carlitz, L. A determinant connected with Fermat's last theorem. Proc. Amer. Math. Soc. **10** (1959), 686–690.

Let Δ_n be the $n \times n$ determinant having the entry in row i and column j equal to $C_{n,k}$ where $0 \leq k \leq n-1$, $k \equiv n-i+j \pmod n$. The author shows that

$$\Delta_{p-1} \equiv p^{p-2} \prod_{a=1}^{p-2} [(1+a)q(1+a)-aq(a)] \pmod{p^{p-1}},$$

where $q(a)=(a^{p-1}-1)p^{-1}$, p not a divisor of a.

R. G. Stanton (Waterloo, Ont.)

7183:

van der Blij, F. An invariant of quadratic forms mod 8. Nederl. Akad. Wetensch. Proc. Ser. A **62**=Indag. Math. **21** (1959), 291–293.

Let $f(x, y)$ be an integral quadratic form in n variables with odd determinant D and signature τ. If the integral vector w satisfies the congruence $f(x, x) \equiv f(x, w) \pmod 2$ for all integral vectors x, then

$$(*) \qquad \exp\{\tfrac{1}{4}\pi i(f(w, w)-\tau)\} = |D|^{1/2} G_0^{-1},$$

where $G_0=\sum_{y \in M'/M} \exp 4\pi i\{f(y, y)\}$, M is the lattice of all integral vectors and M' is the dual lattice relative to f. This generalizes a formula due to the reviewer [Comment. Math. Helv. **33** (1959), 34–37; MR **20** #6999]. Furthermore, the author shows that the right-hand side of (*)

can be expressed in terms of characters of the Witt group. For a unimodular form the formula reduces to $f(w, w) \equiv \tau \pmod 8$, a result which is related to recent topological work by Hirzebruch and H. Hopf [Math. Ann. **136** (1958), 156–172; MR **20** #7272]. *W. Ledermann* (Manchester)

7184:

Springer, T. A. On the equivalence of quadratic forms. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 241–253.

The author describes the Hasse invariant of a quadratic form by a new homological-algebraic method. Let Q be a quadratic form over a field K. He shows first that there is a 1-1 correspondence between the set of equivalence classes, over K, of forms which become equivalent to Q in an extension field L/K, and the noncommutative 1-cohomology classes of the orthogonal group of Q (in L), over the Galois group of L/K. Using the Clifford algebra he derives from this the Hasse invariant.

By the same methods he handles nondefective quadratic forms over fields of characteristic 2, obtaining results of C. Arf [Rev. Fac. Sci. Univ. Istanbul Sér. A 8 (1943), 297–327; MR 7, 359, 621].

G. Whaples (Bloomington, Ind.)

7185:

Stuchlik, Franz. Beiträge zur Zahlentheorie Hermitescher Formen. I. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **2** (1958), 11–14.

The author studies binary hermitian forms:

$$h = h(x, x'; y, y') = axx' + (b'/2)xy' + (b/2)x'y + cyy',$$

in which a and c are integers of the rational field k, and b, b', and the variables x, x' and y, y' are conjugate pairs of integers of $k(i)$. He calls the rational integer $bb' - 4ac = D$ the discriminant of h, and calls D a "stem-discriminant" [cf. Brandt, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Nat. Kl. **99** (1951), no. 1; MR **13**, 537] if there does not exist a linear transformation $x_1 = \alpha x + \beta y$, etc., with integral α and β in $k(i)$, which yields h when applied to a form of smaller discriminant in absolute value than D. The author proves that all rational integers $\not\equiv 3 \pmod 4$ are discriminants of forms like h, while stem-discriminants are square-free, not divisible by primes $\equiv 1 \pmod 4$, have $D_1 \equiv 3 \pmod 4$ if $D = 2D_1$, and are products of prime stem-discriminants. His other results deal with the equivalence of forms and the existence of representations of rational integers by forms of a prescribed stem-discriminant. In the latter case, he employs the Dirichlet theorem on primes in arithmetical progression. For each of his results, the author states the analog for quadratic forms $ax^2 + bxy + cy^2$, a, b, c, x and y integral in k. *R. Hull* (Santa Monica, Calif.)

7186:

Shanks, Daniel. Quadratic residues and the distribution of primes. Math. Tables Aids Comput. **13** (1959), 272–284.

The author gives data and discussion about anomalous distribution phenomena of primes, mostly primes under 3 million.

One of these phenomena is the preponderance of the number of primes of the form $4n-1$ over those of the form $4n+1$ noticed by Chebyshev. Let $\Delta(x)$ denote the excess of the number of primes of the former form $\leq x$ over the latter so that

$$\Delta(x) = -\sum_{2<p\leq x} (-1)^{(p-1)/2}.$$

Although $\Delta(26861) = -1$ and $\Delta(x)$ is negative again just beyond 617000 a census to $x = 3.10^8$ shows that 99.84 percent of the integers n make $\Delta(n) > 0$. The author believes that this preponderance is not a temporary phenomenon and makes the following conjecture. Let $\pi(x)$ denote the number of primes $\leq x$. Define $\tau(n) = \Delta(n) n^{1/2}/\pi(n)$; then

$$\sum_{n\leq x} \tau(n) \sim x.$$

The closest result to this conjecture is the existence of a constant $K > 0$ such that the above sum exceeds Kx for all sufficiently large x. This follows from theorems of Hardy, Littlewood and Landau but involves the assumption that the associated L-series has no zero with real part greater than $\frac{1}{4}$. The overall deficiency of primes of the form $4m+1$ appears to be due to the scarcity of primes $8m+1$, the other three kinds of primes being roughly equally popular. Similarly primes $12m+1$ are scarce as compared to the equally numerous primes of the forms $12m+5, 7, 11$.

On the other hand the 4 classes of primes modulo 10 are equally distributed. These facts are explained by considering the multiplicative group of residue classes prime to M. For $M = 8$ and 12 the group is the fours group whereas for $M = 10$ it is the cyclic group. In general, primes of the form $Mk + r$ will be less numerous when r is a quadratic residue of M.

The paper closes with some remarks about the analogous phenomenon $\pi(x) < \mathrm{Li}\,(x)$ which fails for infinitely many unknown values of x. The author defines $\rho(n) \equiv \{\mathrm{Li}\,(n) - \pi(n)\} n^{1/2}/\pi(n)$ and conjectures that

$$\sum_{n\leq x} \rho(n) \sim x.$$

D. H. Lehmer (Berkeley, Calif.)

7187:

Clark, Robert E.; und Rohrbach, Hans. Zur Theorie der asymptotischen Dichte. II. J. Reine Angew. Math. **201** (1959), 113–118.

[For part I, see H. Rohrbach and B. Volkmann, same J. **192** (1953), 102–112; MR **15**, 506.] Let $A, B, \cdots$ be sets of non-negative integers, $A(x)$ the number of positive integers in A which do not exceed x. Put $\liminf (A(x)/x) = \Delta^*(A)$, $\liminf [(A(x) + B(x))/x] = \sigma^*(A, B)$, $I(x) = x - A(x) - B(x)$, $K(x) = \max\{I(0), I(1), \cdots, I(x)\}$. If $I(h-1) = K(h-1)$, if $I(h) < K(h)$ and if h' is the smallest integer exceeding h for which $I(h') = K(h')$, then the sequence $h, h+1, \cdots, h'$ is called a principal interval. The author's main result is a proof of the following new form of a theorem of Ostmann: If $0 \in A \cap B$, then either $\Delta^*(A+B) \geq \min[1, \sigma^*(A, B)]$ or the length of the principal intervals is bounded and there are infinitely many principal intervals of maximal length m and $\Delta^*(A+B) \geq (m-1)/m^{-1}\sigma^*(A, B)$.

Other theories of Ostmann can be quickly obtained from the authors' result. In particular, if $0 \in A \cap B$, if j is the smallest integer for which $I(j) > 0$ and if $\sigma^*(A, B) \leq 1$, then $\Delta^*(A+B) \geq j(j+1)^{-1}\sigma^*(A, B)$.

H. B. Mann (Columbus, Ohio)

7188:

Rieger, G. J. Über Partitionen. Math. Ann. **138** (1959), 356–362.

Let $p_k(n)$ denote the number of partitions of n into k positive summands. The author sharpens results of Iseki and of Gupta by proving:

$$p_k(n) = \frac{n^{k-1}}{k!(k-1)!}\left(1+\frac{k(k-1)(k-3)}{4n}+O(n^{-2})\right)$$

for fixed $k \geq 3$ and n large. Also, for all k, $n \geq 1$,

$$p_k(n) \leq \frac{1}{k!(k-1)!}\left(n+\frac{k(k-3)}{4}\right)^{k-1}+1.$$

If n runs through a residue class mod $k!$, then $p_k(n)$ is a polynomial of degree $k-1$. The coefficients are given explicitly in terms of Bernoulli numbers.

N. J. Fine (Princeton, N.J.)

7189:

Iseki, Shô. A partition function with some congruence condition. Amer. J. Math. **81** (1959), 939–961.

Let $p(n; a, q)$ be the number of partitions of the integer n into parts congruent to $\pm a \pmod q$ (q prime > 3). Livingood [Amer. J. Math. **67** (1945), 194–208; MR **6**, 259] found a representation of $p(n; a, q)$ by a convergent series, using the Hardy-Ramanujan method of contour integration, with the modifications of Rademacher [Proc. London Math. Soc. (2) **43** (1937), 241–254; Amer. J. Math. **60** (1938), 501–512] and the technique applied by Lehner [Duke Math. J. **8** (1941), 631–655; MR **3**, 166] to the particular case $p=5$. The author now solves this problem for the case of an arbitrary (not necessarily prime) modulus M. The method is still that of Hardy-Ramanujan-Rademacher; also some of the techniques of Lehner and Livingood are used, but new difficulties arise, because of the composite modulus. These are overcome, essentially by the following device. For integral h, k, $(h, k) = 1$, $k \geq 1$, one sets $k = k_1 D$, $M = m_1 D$, $(k_1, m_1) = 1$, $K = k_1 m_1 D$, so that $D = (k, M)$, $K = \text{l.c.m. } (k, M)$; one also selects arbitrary integers γ, δ satisfying $\gamma k_1 - \delta m_1 = 1$, and denotes by H any fixed solution of $hH \equiv \delta \pmod k$. It turns out that most of the Lehner-Livingood results carry over, provided that one uses D instead of p (if $p|k$) or of 1 (if $p\nmid k$) and K instead of k (if $p|k$) or of pk (if $p\nmid k$). This introduces actually more symmetry into the formulae than can be found in older papers; the essential dichotomy ($D=1$, $D>1$), however, occurs again. Set $x = \exp(2\pi i h/k - 2\pi z/k)$, $x' = \exp(2\pi i H/k - 2\pi/Kz)$ ($\Re z > 0$), $b = ha - D[ha/D]$, $\rho = \exp(-2\pi i \gamma a/M)$, $A = 6a^2 - 6Ma + M^2$, $B = 6b^2 - 6Db + D^2$ and

$$F(x'; b, D, \rho) = \prod_{m \geq 0} (1-\rho x'^{mD+b})^{-1}(1-\rho^{-1}x'^{mD+D-b})^{-1};$$

then generating function of $p(n; a, M)$ is

$$F(x; a, M) = \omega(h, k)\exp\{(\pi/6Mk)(B/z - Az)\}F(x'; b, D, \rho),$$

$|\omega(h, k)| = 1$, as follows from the author's functional equation [Duke Math. J. **24** (1957), 653–662; MR **19**, 943]. From here on the proof follows the customary lines, with the reduction of exponential sums to generalized Kloosterman sums (evaluated non-trivially following Salié or Weil) and the identification of loop integrals with Bessel functions. The result, of course, is a generalization of Livingood's formula, to which it reduces for $M = p$. [See also Petersson, Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Nat. **1954**, no. 2; MR **17**, 129; thm. 13.] {The reviewer would like to point out a surprising similarity in technique with that used by P. Hagis [Dissertation, Univ. of Pennsylvania, 1959, unpublished] to obtain a generalization of Livingood's result in an entirely different direction, namely the establishment of a convergent series representation for $p(n; a_1, a_2, \cdots, a_r; q)$, the number of partitions of n into parts congruent (mod q) to any of the integers $\pm a_1, \pm a_2, \cdots, \pm a_r$ ($r \leq \frac{1}{2}(q-1)$, q prime).}

E. Grosswald (Princeton, N.J.)

7190:

Starčenko, L. P. The construction of a completely uniformly distributed sequence. Dokl. Akad. Nauk SSSR **129** (1959), 519–521. (Russian)

The author gives an example of a completely uniformly distributed sequence. Let $\{\alpha\}$ be the fractional part of α. Put $n_k = [1+\exp(k^3)]$ ($k = 1, 2, 3, \cdots$); also denote by $p_1 = 2, p_2 = 3, \cdots$ the successive primes. Then the sequence is as follows: $\{\log 2\}, \{2\log 2\}, \cdots, \{n_1 \log 2\}$; $\{\log 2\}, \{\log 3\}, \{2\log 2\}, \{2\log 3\}, \cdots, \{n_2 \log 2\}, \{n_2 \log 3\}; \cdots;$ $\{\log p_1\}, \{\log p_2\}, \cdots, \{\log p_r\}, \{2\log p_1\}, \{2\log p_2\}, \cdots,$ $\{2\log p_r\}, \cdots, \{n_r \log p_1\}, \{n_r \log p_2\}, \cdots, \{n_r \log p_r\}; \cdots$.

K. Mahler (Manchester)

7191:

Lehmer, Emma. On Euler's criterion. J. Austral. Math. Soc. **1** (1959/61), part 1, 64–70.

The author obtains the following results. I. If $p = 4n+1 = a^2+b^2$, $a \equiv 1 \pmod 4$, then

$$\begin{aligned} 2^{(p-1)/4} &\equiv (-1)^{b/4} \pmod p \quad (b \equiv 0 \pmod 4), \\ &\equiv a^{-1}b \pmod p \quad \text{(otherwise)}. \end{aligned}$$

II. If $p = A^2+3B^2$, $A \equiv B \equiv 1 \pmod 3$, and 2 is not a cubic residue, then

$$\begin{aligned} D^{(p-1)/3} &\equiv 1 \pmod p \qquad (D \equiv m^3), \\ &\equiv (-A+3B)/2A \quad (D \equiv 2m^3), \\ &\equiv -(A+3B)/2A \quad (D \equiv 4m^3). \end{aligned}$$

III. The residue of $2^{(p-1)/5}$ is expressed in terms of u, v, w, x, where

$$\begin{aligned} 16p &= x^2+50u^2+50v^2+125w^2, \\ xw &= v^3-u^3-4uv, \quad x \equiv 1 \pmod p. \end{aligned}$$

L. Carlitz (Durham, N.C.)

7192:

Newman, Morris. Modular forms whose coefficients possess multiplicative properties. Ann. of Math. (2) **70** (1959), 478–489.

Let $B(\tau) = \eta^r(\tau)\eta^s(q\tau)$, where $\eta(\tau)$ is Dedekind's modular form, r and s are non-zero integers of the same parity and q is a prime. Put $t = (r+sq)/24$, $t^* = (s+rq)/24$, and let p be a prime greater than 3 such that $t(p-1)$ and $t^*(p-1)$ are integers. Let $g(\tau) = B(p\tau)/B(\tau)$ and

$$G(\tau) = \sum_{k=0}^{p-1} g\{\tau/(1-kq\tau)\} + g\{(\tau-1)/(p_0 p - q_0 q\tau)\},$$

where $p_0 p - q_0 q = 1$. The functions $g(\tau)$ and $G(\tau)$ are entire modular functions for the groups $\Gamma_0(pq)$ and $\Gamma_0(q)$. The main result proved is that, if $0 \leq \delta = t(p-1) < p$ and $0 \leq t^*(p-1) < p$, then $G(\tau)$ is a constant. This implies that

$$(*) \qquad c(np+\delta) = \beta c(n) - \gamma p^{s-1} c\{(n-\delta)/p\},$$

where β and γ do not depend on n and $c(m)$ is the coefficient of x^m in the product

$$\prod_{n=1}^{\infty} \{(1-x^n)^r(1-x^{sn})^s\}.$$

These inequalities are satisfied by 147 different sets of values q, r, s; for each triple of values the prime q cannot exceed 23 and the prime p must satisfy a congruence $p \equiv 1 \pmod m$, where m is a certain divisor of 24. If $d(n) = c\{t(n-1)\}$, in the case when $\delta > 0$, (*) implies that

$$d(np) = d(n)d(p) - \gamma p^{s-1} d(n/p),$$

whenever n is composed only of primes $p \equiv 1 \pmod m$. This gives rise to a 'modified Euler product' for the associated Dirichlet series. There is a slightly simpler result for $\delta = 0$.

{These results suggest that there are relationships between the author's work and the general theory of T_n-operators as developed for arbitrary subgroups of the modular group by, for example, K. Wohlfahrt [Math. Nachr. **16** (1957), 233–256; MR **16** #5618] and in the case of the principal congruence group by Petersson [Math. Ann. **117** (1939), 39–64; MR **1**, 294]; these relationships and their generalizations are worth exploration.}

R. A. Rankin (Glasgow)

7193:

Lomadse, G. Über die Darstellung der Zahlen durch einige quaternäre quadratische Formen. Acta Arith. **5** (1959), 125–170.

The author considers the number $r(n)$ of representations of a positive integer n in the form $a(x_1^2+x_2^2)+a'(x_3^2+x_4^2)$, where a, a' are fixed positive integers with $(a, a') = 1$ and x_1, x_2, x_3, x_4 are any integers. Then $r(n)$ is the Fourier coefficient in the expansion of a product of theta-functions

$$F(\tau) = \vartheta_{00}^2(\tau; 0, 2a)\vartheta_{00}^2(\tau; 0, 2a')$$

which is a modular form of dimension -2 belonging to the principal congruence group $\Gamma(4aa')$. As in the classical investigations of Hardy, Littlewood and Mordell, the form of this function at rational points τ is investigated and yields a singular series $\Theta(\tau; aa')$, which is itself a modular form of dimension -2 for $\Gamma(4aa')$; $\Theta(\tau; aa')$ is, of course, a combination of Eisenstein series which behaves similarly to $F(\tau)$ at the parabolic cusps of the fundamental region and the difference $F(\tau) - \Theta(\tau; aa')$ is a cusp form which is found explicitly in the cases

$$a = 1, a' = 5, 6, 7, 9, 10; \quad a = 2, a' = 5.$$

This yields formulae for $r(n)$. For example, when $a = 1$, $a' = 5$, the cusp form is $\frac{1}{4}\vartheta_{21}^2(\tau; 0, 10)\,\vartheta_{21}^2(\tau; 0, 10)$, while, when $a = 2$, $a' = 5$, it is a linear combination of three such products of theta-functions.

The details of the analysis are very complicated and employ methods and results due to Kloostermann, Streefkerk and the author. See, in particular, H. Streefkerk, Thesis, Free Univ. of Amsterdam, 1943 [MR **7**, 414] and G. A. Lomadze [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 77–102; MR **19**, 942].

When $a = 1$, $a' = 5, 6, 7, 9$ the results agree with those stated by Kloostermann and Liouville.

R. A. Rankin (Glasgow)

FIELDS

See also 7221, 7222.

7194:

O'Meara, O. T. Infinite dimensional quadratic forms over algebraic number fields. Proc. Amer. Math. Soc. **10** (1959), 55–58.

The author proves that nondegenerate quadratic forms in a countably infinite number of variables are equivalent over an algebraic number field F if and only if they are equivalent over all real completions of F. Main tools: The "weak approximation theorem" of product formula fields, and Witt's theory of finite-dimensional quadratic forms. *G. Whaples* (Bloomington, Ind.)

7195:

Leopoldt, Heinrich-Wolfgang. Über die Hauptordnung der ganzen Elemente eines abelschen Zahlkörpers. J. Reine Angew. Math. **201** (1959), 119–149.

Let K be an abelian field over the rational field R. Let R_f be the field $R(Z_f)$, where Z_f is a primitive fth root of unity such that $K \subset R_f$ and f is minimal for this property. Then f is the conductor (Führer) of K, and K corresponds as a class-field to a subgroup H of the multiplicative group G of relatively prime residues modulo f in R, such that the factor group G/H is isomorphic to the galois group g of K over R. The field K has the algebraically normal basis over R consisting of the period $\eta = \sum_{v \in H} Z_f^v$ and its conjugates. If f is square-free, this basis is also a minimalbasis (Ganzheitsbasis) for the ring I (Hauptordnung) of all integral elements of K with respect to the rational integers. If f is not square-free there are subfields of R_f for which normal minimalbases do not exist, namely, those in which higher or irregular ramification occurs. In such cases the periods do not readily lead to minimalbases, or at least not to such which have invariant or simple behavior with respect to g. By a different approach, minimalbases having these desirable properties are determined for any K in the present paper. To do this, the author exploits the operator isomorphism of K/R as a g-module with the group ring

$$\mathfrak{G} = R \cdot g \sim K = \mathfrak{G}A = \sum_{\sigma \in g} R \cdot A^\sigma,$$

where A is any g-regular element of K, that is, one whose conjugates are linearly independent over R, e.g., η above. His main result is that $I = O_K T_K$, where O_K is a uniquely determined sub-module of $\mathfrak{G}$, and T_K is a uniquely determined g-regular integral element of K. In the course of obtaining this result, the author first proves that the elements of K have a unique representation:

$$A = (1/(K:R)) \sum_{\chi} y_K(\chi|A)\tau(\chi),$$

where χ ranges over the characters of K, i.e., of g, and $\tau(\chi)$ is the normalized proper Gauss sum belonging to χ [see Hasse, *Vorlesungen über Zahlentheorie*, Springer-Verlag, Berlin, 1950; MR **14**, 534; pp. 422–445]. He obtains necessary and sufficient conditions in terms of the invariant coordinates $y_K(\chi|A)$ for A to be g-regular and also for A to be integral. Next the author calls two characters similar if their conductors are divisible by exactly the same powers of all primes which divide either

of the conductors to a power greater than unity. He then calls the sums of sets of similar characters the branch characters of K and uses these to define the above "Basiszahl" T_K and the ring O_K. Before proving the uniqueness of the latter, he proves several theorems about abelian fields, including the above statement about the non-existence of normal-minimalbases. In an Appendix, he discusses higher ramification and relates a series of character groups to the Hilbert ramification group series. *R. Hull* (Santa Monica, Calif.)

7196:

Fröhlich, A. The rational characterization of certain sets of relatively Abelian extensions. Philos. Trans. Roy. Soc. London. Ser. A **251** (1959), 385–425.

Let K be an abelian field of a prime power degree l^n over the rationals P, and let $C(K/P)$ be the set of all cyclic extensions Λ of degree l or 1 over K which are normal over P. The aim of the paper is to give a characterization of the members Λ of $C(K/P)$ in terms of data in P. Thus, let $\overline{\Lambda}$ be the composite of all Λ. Let Φ, $\overline{\Phi}$ be the character groups of the Galois groups Γ, $\overline{\Gamma}$ of K/P, $\overline{\Lambda}/K$, respectively, and let Ψ be the character group of the Galois group of the composite of K and all cyclic fields of degree l over P. Let $\bar{F}(\Gamma, E)$ be the group of 2-cohomology classes of Γ in the group E of complex lth roots of 1 acted on by Γ trivially. Each $\Lambda \neq K$ in $C(K/P)$ determines, since the Galois group $\Gamma(\Lambda/K)$ of Λ/K lies in the center of that of Λ/P, $l-1$ (not necessarily distinct) elements of $\bar{F}(\Gamma, E)$ corresponding to $l-1$ isomorphisms of $\Gamma(\Lambda/K)$ with E, and the totality of such elements of $\bar{F}(\Gamma, E)$ associated with some members of $C(K/P)$ forms a subgroup $A(K)$ of $\bar{F}(\Gamma, E)$ when 1 is added. Now, one gets an exact sequence $1 \to \Phi \to \Psi \to \overline{\Phi} \to A(K) \to 1$. This reduces the above problem to that of determining $A(K)$, and it is proved that an element of $\bar{F}(\Gamma, E)$ lies in $A(K)$ if and only if its representative cycle c satisfies the following conditions: (i) if p divides the conductor f of K/P and $p \equiv 1 \bmod l$, then

$$\prod_{r \bmod p-1} c_p(a, a^r) = c_p(a, p)c_p(p, a)^{-1}$$

for all rational idèles a which are units at p, and (ii) if $2|f$, then

$$\prod_{r \bmod 2} c_2(-1, (-1)^r) = c_2(5, 2)c_2(2, 5)^{-1},$$

where $c_p(a, b)$ stands for $c((a, K/p), (b, K/p))$ (but is described also differently). For the proof, everything is decomposed into two parts corresponding to the ramification of F, G respectively, where F, G are respectively the set of primes ramified or unramified in K, and an essential step in such a decomposition is carried out by the use of the author's former work [Proc. London Math. Soc. (3) **4** (1954), 235–256; MR **16**, 116]. Besides some alternative forms of the above conditions (i), (ii), also a more explicit criterion for $A(K)$ and a description of $\overline{\Phi}$ are given. Relation with similar local invariants is discussed and ramification in each Λ is studied, while a fuller discussion of decomposition law is reserved for a subsequent paper. The former half of the paper, which includes e.g. the above exact sequence, deals with the case where P is quite general (instead of being the rationals), and a generalization of the above characterization of $A(K)$ to the case of a general algebraic number field P is promised for a future publication.

T. Nakayama (Nagoya)

7197:

★Krull, Wolfgang. Elementare und klassische Algebra. II. Sammlung Göschen Bd. 933. Walter de Gruyter & Co., Berlin, 1959. 132 pp. DM 3.60.

This second volume is primarily an exposition of Galois theory. After the gentle introduction to abstract algebra in volume I [de Gruyter, 1952; MR **14**, 442] the author feels free to indulge in the abstractions of "field" and "group". He devotes more space than is now customary in such expositions to questions of computability, and works out in detail the "bicyclic" case. In discussing the unsolved question of the existence of normal extension of the rational field with prescribed Galois group, he mentions without proof Hilbert's irreducibility theorem, and states a "conjecture" the verification of which would prove the higher-dimensional analog of Lüroth's theorem. In point of fact, a counterexample to the conjecture has been given by F. Enriques [Atti Accad. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (5) **21** (1912), no. 1, 81–83] for the case of dimension 3. *E. R. Kolchin* (New York, N.Y.)

7198:

Latt, Klaus. Über die Galoisgruppe des in bezug auf den Körper der rationalen Zahlen maximalen abelschen Körpers. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **7** (1957/58), 583–608. (Russian, English and French summaries)

Using the fact that the field A of all roots of unity over the rational number field is the union of the fields T_p, each the union of the fields of p^nth roots of unity, $n = 1, 2, \cdots, p$ ranging over the set of all rational primes, the author examines the automorphism group of A by means of the simpler structure of the automorphism groups of the fields T_p. The closed subgroups of these groups are described, as are their character groups.

O. F. G. Schilling (Chicago, Ill.)

7199:

Michler, Lothar. Zusammenhang zwischen dem Isomorphismengruppoid eines algebraischen Erweiterungskörpers und der Galoisgruppe des Galoisschen Oberkörpers. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **2** (1958), 1–3.

Let Ω be a separable normal extension of a field K_0 (the ground field), and let Σ be a finite algebraic extension of K_0. Then $\Sigma = K_0(\theta)$, where θ is a root of a unique monic, separable and irreducible polynomial $f(x) \in K_0[x]$, of degree n, say. Let $\theta_1 (= \theta), \theta_2, \cdots, \theta_n$ be the roots of $f(x)$ in Ω. The fields

$$(1) \qquad (\Sigma =) K_0(\theta_1), K_0(\theta_2), \cdots, K_0(\theta_n)$$

comprise a complete set of conjugate fields of Σ, but these fields need not be distinct. If exactly r of them are distinct, then r divides n and we write $n = rg$.

Now the K_0-isomorphisms and automorphisms between all pairs of fields (1) form a Brandt-groupoid Γ of rank r and order g. Generally, with such a groupoid there is associated a group $\mathfrak{L}(\Gamma)$, the Loewy group of Γ, of order $g^r n!$. On the other hand, we may consider $K = K_0(\theta_1, \theta_2,$

$\cdots, \theta_n)$, which is the least normal extension of K_0 containing Σ. The author proves that the Galois group of K is isomorphic with a subgroup of $\mathfrak{T}(\Gamma)$.

W. Ledermann (Manchester)

7200:

Demuškin, S. P. The group of the maximum p-extension of a local field. Dokl. Akad. Nauk SSSR **128** (1959), 657–660. (Russian)

Let the composite of all finite normal p-extensions of the local field Ω have the group G relative to Ω; let $p \neq 2$; let ν be two more than the degree of Ω over the local rational field; let $q = p^m$ (with $m \geqq 1$) be the maximal power of p such that Ω contains a qth root of unity. Then G is isomorphic to the group with ν generators $\sigma_1, \sigma_2, \cdots, \sigma_\nu$ and the single relation

$$\sigma_2^q[\sigma_1, \sigma_2][\sigma_3, \sigma_4] \cdots [\sigma_{\nu-1}, \sigma_\nu] = 1.$$

The paper extends work of Kawada [J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1954), 1–18; MR **16**, 6].

R. A. Good (College Park, Md.)

7201:

Zervos, Spiros. Espaces vectoriels sur un corps commutatif totalement ordonné; applications. C. R. Acad. Sci. Paris **248** (1959), 1762–1764.

In Bourbaki, Book V, linear topological spaces on non-discrete, valued (valué) fields are treated. The author of the present note generalizes Bourbaki's treatment by introducing linear topological spaces on non-discrete K_0-valued fields, where K_0 is an arbitrary totally ordered field with the order topology. It is stated that a large portion of the existing theory (for K_0 = the field of real numbers) on linear topological spaces, including the separation theorems, is again valid for a general K_0. The detail was scheduled to appear in *Séminaire Dubreil, Dubreil-Jacotin, Pisot; 12e année*:1958/1959, Secrétariat mathématique, Paris, 1960, but has been listed there only by title. The author mentions three new theorems and indicates their applications.

I. Namioka (Ithaca, N.Y.)

ALGEBRAIC GEOMETRY

See also 7221, 7222, 7253, 7517.

7202:

★Chevalley, Claude. Fondements de la géométrie algébrique. Secrétariat Mathématique, Paris, 1958. 222 pp. (mimeographed)

This is a course in the point-set topology of algebraic varieties, self-contained except for three pages of statements from algebra. The language of the Zariski topology (no sheaves) is used systematically, everything being done over a fixed algebraically closed ground field. There are no generic points, their place being taken by various considerations involving function fields. The following chapter headings, each followed by an outline of the contents, will give an idea of what is covered: The Zariski topology (definition of affine and abstract varieties, irreducibility); Functions (functions from one variety into another, subvarieties, products, constructible sets—these latter are finite unions of intersections of open and closed subsets of a variety, and are closed under the lattice operations, products, and projections); Dimension (definition, dimension of intersections, systems of parameters, constructibility of sets associated with a rational map and a dimension or irreducibility condition, the irreducibility of the general hyperplane section); Complete varieties (proper correspondences and maps—the latter being the analogues of proper maps in topology, defined by insisting that inverse images of compact sets be compact—complete varieties, coverings); Normal varieties (definition, Zariski's main theorem, normalization of a variety in a larger field, applications to the local study of a rational map—e.g., an everywhere defined everywhere finite-to-one map from a variety into a normal variety of the same dimension is open); Simple points (the tangent space at a point, the local behavior of a map at a non-ramified point—i.e., a point at which the corresponding map of tangent spaces is injective—simple points and uniformizing parameters).

M. Rosenlicht (Berkeley, Calif.)

7203:

Wallace, Andrew H. Differentiation of algebraic functions. Proc. London Math. Soc. (3) **9** (1959), 465–480.

Let $z_1, \cdots, z_m$ be separably algebraic over the field $k(x_1, \cdots, x_n)$, where the x_i are independent indeterminates over an arbitrary field k. The author defines the derivatives of the z_j with respect to the x_i in $k(x, z)$ at a specialization (x', z') of (x, z). Various results are established, the most important of which are the rule for differentiation of a function of a function, the implicit function theorem, and Taylor's theorem.

It seems that these results have not previously been brought together in this general form, even when k has characteristic zero. This is surprising, as special cases are frequently required in algebraic geometry. Thus, although the results are not completely unknown, the author has filled a gap in the literature.

D. Kirby (Leeds)

7204:

Godeaux, Lucien. Construction d'une surface projectivement canonique. Bull. Soc. Roy. Sci. Liège **28** (1959), 165–169.

If a V_3^{2r-4} in S_r (algebraic variety of three dimensions and order $2r-4$ in complex projective r-dimensional space) has as its general hyperplane section a surface with all genera equal to unity, and thus as its general S_{r-2} section a canonical curve of genus $r-1$, it is known that the general quadric section of V_3^{2r-4} is the canonical model of a surface of genera $p_g = p_a = 2r-5$, $p^{(1)} = 4r-7$ [F. Enriques and M. L. Campedelli, *Lezioni sulla teoria delle superficie algebriche*, Cedam, Padova, 1932; pp. 331 ff.]. The present paper is devoted to pointing out that for $r = 6$ one possible form of V_3^8 is the cubic section, residual to one generating S_3, of the cone projecting from a point outside its ambient S_5 the rational V_3^3 generated by ∞^1 planes in S_5.

P. Du Val (London)

7205:

Godeaux, Lucien. Note sur une transformation birationnelle de l'espace. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 432–440.

The author discusses a 3-5 Cremona transformation in 3-space. A detailed description of this transformation can be found in the paper by E. Caporali [Ann. Mat. Pura Appl. (2) **7** (1875), 149–188].

A. Gutwirth (Berkeley, Calif.)

7206:

Turri, Tullio. Sui tipi di trasformazioni birazionali involutorie negli iperspazi. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 9–26.

The author develops the theory in S_r, where $r>3$, of involutorial birational transformations of the following types: (A) de Jonquières involutions; (B) transformations constructed by the polarity with respect to r quadrics; and (C) transformations constructed by means of linear systems of order $r+1$ and dimension r and intersecting themselves, r by r, outside of a fixed curve, in a pair of points. For $r=2, 3$, there exist involutorial transformations not of these types nor the product of permutable involutorial transformations of these types. The question is raised as to whether such transformations can exist for $r>3$.

G. B. Huff (Athens, Ga.)

7207:

Turri, Tullio. Trasformazioni cicliche in S_r. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 27–32.

The author shows that a cyclic birational transformation in S_r is birationally equivalent to (i) a non-homographic involution; (ii) a central cyclic homology; (iii) the product of permutable central cyclic homologies; or (iv) the product of a non-homographic involution I_2 with central cyclic homologies which are permutable among themselves and with I_2.

G. B. Huff (Athens, Ga.)

7208:

Zariski, Oscar. Proof that any birational class of non-singular surfaces satisfies the descending chain condition. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 21–31.

The author gives here another proof, for algebraic surfaces, of the proposition III.1.3 of his memoir *Introduction to the problem of minimal models* [Math. Soc. Japan, Tokyo, 1958; MR **20** #3872]. A birational correspondence T: $V\to V'$ is called antiregular when T^{-1} is a regular one. If T is antiregular, V' is said to dominate V ($V\succ V'$). The above-mentioned theorem says: Every strictly descending chain $F\succ F_1\succ\cdots$ of birationally equivalent non-singular surfaces is finite.

The proof, ad absurdum, goes along the following way: If P_n is a fundamental point of $F_n\to F_{n-1}$ and $\mathscr{E}_n$ is the cycle corresponding to P_n in $F_n\to F$, it is verified (Th. 1) that the set $\{\mathscr{E}_i\}$ contains an infinite subset $\{\mathscr{E}_{i_j}\}$ of maximal elements of the set. One can fix an irreducible exceptional curve E_j for each $\mathscr{E}_{i_j}$ and it is proved that: (1) $E_j\cap E_k=\emptyset$; (2) $(D\cdot E_j)=1$ for all j, where D is a cycle of the anticanonical system $|-K|$ on F and the left-hand side of (2) denotes the intersection index of D and E_j; (3) the relation (2) holds for all D of a certain linear subsystem L of $|-K|$, hence E_j is a component of one and only one cycle D_j of L; (4) $(E_j^2)=-1$. The propositions (3) and (4) are in contradiction.

P. Abellanas (Madrid)

7209:

Nering, Evar D. On extensions of the field of constants of an algebraic function field. Proc. Amer. Math. Soc. **10** (1959), 198–202.

The purpose of this paper is to discuss the applications of the formula

$$g-1=\sum_{i=1}^{s} r_i\cdot e(H_i)(g_i-1)+\rho$$

which was proved by the author in Ann. of Math. (2) **67** (1958), 590–606 [MR **20** #4542]. First, the author, using this formula, gives a proof of the theorem which asserts that the genus of an algebraic function field K of one variable with constant field k remains unchanged when a transcendental element is adjoined to K and k. Then the case when the adjoined element is separable is considered and the author proves that, in his formula, $s=1$, $r_1=1$ and $\rho=0$, and consequently the genus still remains unchanged. Finally the inseparable case is analyzed and the author uses a result by J. Tate [Proc. Amer. Math. Soc. **3** (1952), 400–406; MR **13**, 905] in order to prove that ρ is divisible by $(p-1)/2$.

E. Lluis (México, D.F.)

7210:

Horrocks, G. A Hilbert function for stacks on Grassmann varieties. Proc. London Math. Soc. (3) **9** (1959), 51–73.

Soit G la grassmannienne des K^m de K^N, $m\leqq N-m=n$, K un corps algébriquement clos de caractéristique quelconque. Le sous-espace de $G\times K^N$ des points $x\times e$ tels que le point e de K^N appartient au K^m déterminé par le point x de G, est un fibré de fibre K^m, base G et groupe $\mathrm{GL}(m)$. Soit V l'espace fibré contragrédient à celui-ci et $V(\lambda)$ le fibré avec la même base et le même groupe et ayant comme fibre l'espace de représentation correspondant à une partition $(\lambda)=(\lambda_1,\cdots,\lambda_m)$. Soient encore Ω et $\Omega(\lambda)$ les faisceaux des germes de sections locales de V et $V(\lambda)$ respectivement (si $m=1$, $\Omega(\lambda)$ est le faisceau des germes de fonctions rationnelles sur K^N de degré λ). Soit finalement $\mathscr{F}$ un faisceau algébrique cohérent sur G, $\chi(\mathscr{F}\otimes\Omega(\lambda))$ la caractéristique d'Euler-Poincaré de G à valeurs dans le faisceau $\mathscr{F}\otimes\Omega(\lambda)$, et $\{\lambda\}=\{x_1,\cdots,x_m;\lambda\}$ la S-fonction d'arguments $x_1,\cdots,x_m$. On définie $\Upsilon(\mathscr{F})=\sum\chi(\mathscr{F}\otimes\Omega(\lambda))\{\lambda\}$, somme sur toutes les partitions (λ), (généralisation de la fonction génératrice de la fonction d'Hilbert). L'auteur démontre: (1) Si d est la dimension du support du faisceau $\mathscr{F}$ alors $(t_1t_2\cdots t_m)^{-m}\Upsilon(\mathscr{F})$ est un polynôme symétrique en $t_1,\cdots,t_m$ de degré d, où $t_i=(1-x_i)^{-1}$. (2) Si $\mathscr{F}$ est le faisceau d'anneaux locaux d'une sous-variété W de G et W est continuellement équivalente à $\sum c_\lambda[n-\lambda_m,\cdots,n-\lambda_1]$, alors $\mathscr{H}_i(\Upsilon(\mathscr{F}))=(t_1\cdots t_m)^m\sum c_\lambda\{t_1,\cdots,t_m;\lambda\}$, où $[\alpha_1,\cdots,\alpha_m]$ désigne la sous-variété de Shubert de G associée à une chaîne d'espaces vectoriels où l'espace d'indice i est de dimension $N-m+i-\alpha_i$ et $\mathscr{H}_i(P)$ dénote la somme des termes de plus haut degré du polynôme $P(t_1,\cdots,t_n)$.

E. Lluis (México, D.F.)

7211:

★**Chevalley, Claude. La notion de correspondance propre en géométrie algébrique.** Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 152, 11 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

A correspondence Γ between varieties U and V is said to be proper if for all varieties T and all subsets E of $T\times U$ that are closed in the Zariski topology, the set $E\circ\Gamma$ is closed in $T\times V$. (Here the composition $E\circ\Gamma$ is taken in the usual sense of the algebra of relations.) A function f on U with values in V is called proper if the adherence of the graph of f in $U\times V$ is a proper correspondence. This article contains a systematic exposition of the theory of proper correspondences and functions, with applications to the theory of complete varieties. The role of the notion of proper correspondence in the theory of holomorphic functions is elaborated in the following article [see review below].

It is shown that a correspondence Γ between U and V is proper if and only if Γ is closed and the mapping $(t,(x,y))\to(t,y)$ of $T\times\Gamma$ into $T\times V$ is closed for all varieties T. From this it is deduced that Γ is proper if and only if the irreducible components of Γ are proper. If f and g are proper functions, if $g\circ f$ is defined and if $g*f$ is an extension of $g\circ f$ with domain equal to that of f, then $g*f$ is proper. Conditions are given under which one can conclude that f or g is proper from the fact that $g*f$ or $g\circ f$ is proper. A function f on U with values in V is said to be proper at a point y of V if y is adherent to $f(U)$ and there exists an open subvariety V_0 of V that contains y and is such that the restriction of f to $f^{-1}(V_0)$ is proper. A function f is proper if and only if it is proper at each point of the adherence of $f(U)$.

The notion of a complete variety is defined in terms of proper morphisms (rational functions everywhere defined) by the condition that U is complete if there is a proper morphism of U into a variety reduced to a point A. Various properties of complete varieties are derived. Among other results are (a) the theorem of Chow to the effect that any variety V is the image of a complete variety U by a proper function f that is birational and has a closed graph in $U\times V$, in particular, U can be taken as a subvariety of a product of projective lines; and (b) if G is an algebraic group and H is a subgroup such that H and G/H are complete then G is complete. The second of these results is established by showing that the canonical mapping $G\to G/H$ is a proper morphism. In the last section of the paper the author derives the usual algebraic criterion that a variety V is complete if and only if there is a point $y\in V$ for each place π of the function field F_V such that π extends the mapping of the elements of the local ring $\mathfrak{o}(y)$ onto their values at y. (This mapping is called the assignation associated with y and denoted by ζ_y.) An analogous criterion for proper functions is then given as follows. Let g be a function on V with values in a variety W of such a nature that $g(V)$ is dense in W, and let φ be the isomorphism of the field F_W into the field F_V associated with g. If $z\in W$ and ζ_z is the assignation of F_W associated with z, let $\zeta_z{}^{\varphi}$ denote the mapping that assigns to an element of the form $\varphi(w)$ in F_V the value $\zeta_z(w)$. A necessary and sufficient condition that g be proper at z is that for each place π of F_V that extends the mapping $\zeta_z{}^{\varphi}$ there exist a point $y\in V$ such that π extends ζ_y.

H. T. Muhly (Iowa City, Iowa)

7212:

★**Chevalley, Claude. La théorie des fonctions holomorphes de Zariski. Application au théorème de connexité.** Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 158, 17 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

This article gives an exposition of a modified version of Zariski's theory of holomorphic functions. The chief point of difference between the presentation under review and that of Zariski lies in the fact that the invariance theorem is proved here for meromorphic rather than holomorphic functions. While this result is somewhat weaker than Zariski's, it suffices to prove the connectedness theorem for algebraic correspondences, and its proof in these terms is asserted to be technically less difficult than that of Zariski.

In the preliminary sections of the paper, the ring $\mathfrak{R}(U,A)$ of functions on U holomorphic along set A is introduced and the principle of analytic continuation is established. It is shown that if A is closed and $\mathfrak{R}(U,A)$ is an integral domain, then A is connected, and that if A is connected and closed and U is normal then $\mathfrak{R}(U,A)$ is an integral domain. After a section devoted to holomorphic functions on affine varieties, an algebra $\mathfrak{M}(U,A)$ of functions meromorphic on U along A is defined in the following manner. If x is a point of U, $\mathfrak{o}(x)$ is the local ring at x and F_U is the function field on U, let $\mathfrak{M}(x)$ be the product $\mathfrak{o}(x)\otimes F_U$ taken over $\mathfrak{o}(x)$. The ring $\mathfrak{M}(x)$ is identified with the ring of quotients of the completion $\overline{\mathfrak{o}(x)}$ of $\mathfrak{o}(x)$, relative to the multiplicative system S of non-zero elements of $\mathfrak{o}(x)$. If A is a subset of U, a meromorphic function on U along A is a mapping v that assigns to each point x of A an element $v(x)$ in $\mathfrak{M}(x)$ of such a nature that if x_0 is a point of A, there is an open neighborhood U' of x_0 in U and a function $z\in F_U$, $z\neq 0$, such that $zv(x)\in\overline{\mathfrak{o}(x)}$ for all $x\in U'\cap A$ and the mapping $x\to zv(x)$ defines a holomorphic function on U' along $U'\cap A$. The set of such functions forms an algebra $\mathfrak{M}(U,A)$ that contains the ring $\mathfrak{R}(U,A)$. It is proved that if A is a closed subset of U and $\mathfrak{M}(U,A)$ is an integral domain then A is connected, and if U is normal the connectedness of A implies that $\mathfrak{M}(U,A)$ is an integral domain. The invariance theorem now takes this form: "If f is a proper morphism [see review above] of a variety V into a variety U, let A be a closed subset of U and let $B=f^{-1}(A)$. The homomorphism of $\mathfrak{M}(U,A)\otimes F_V$ (over F_U) into $\mathfrak{M}(V,B)$ induced by f is then an isomorphism of the first of these rings onto the second." The proof is carried out by a series of reductions and makes use of a Chow representation of V. Finally, the connectedness theorem is proved in the following form: Let f be a proper morphism of V into a normal variety U such that $f(V)$ is dense in U. Let φ be the homomorphism of F_U into F_V associated with f. If $F_V/\varphi(F_U)$ is a purely inseparable extension (i.e., each element of F_V that is algebraic over $\varphi(F_U)$ is purely inseparable over $\varphi(F_U)$) then for each closed connected subset A of U, the set $B=f^{-1}(A)$ is connected.

H. T. Muhly (Iowa City, Iowa)

7213:

Matsusaka, Teruhisa. On a characterization of a Jacobian variety. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 1–19.

In the first part, the author introduces a symbol which yields a useful criterion for numerical equivalence on an abelian variety. Let X, Y be two cycles of complementary

dimension on an abelian variety A; for a generic u on A, the 0-cycle $X \cdot Y_u$ is defined (Y_u denotes the transform of Y by the translation u); then $S(X \cdot Y_u)$ must be of the form $\alpha u +$ constant, where α is an endomorphism of A. The author puts $\alpha = \alpha(X, Y)$; he shows that α depends only upon the class of X and that of Y for numerical equivalence [the proof is based on his earlier paper, Amer. J. Math. **79** (1957), 53–66; MR **18**, 602] and establishes various properties of this symbol, including the following criteria: (i) if X is a positive non-degenerate divisor on A, then $\alpha(X, Y) = 0$ or $\alpha(Y, X) = 0$ implies that the 1-cycle Y is numerically equivalent to 0; (ii) if X is a non-degenerate curve on A (i.e., a curve no translate of which is contained in a proper abelian subvariety of A), then $\alpha(X, Y) = 0$ or $\alpha(Y, X) = 0$ implies that the divisor Y is numerically equivalent to 0. These results are applied to the proof of the following theorem: let X be a subvariety of codimension 1 of an abelian variety A of dimension n; assume that, for $u_1, \cdots, u_n$ generic and independent on A, the 1-cycle $X_{u_1} \cdots X_{u_{n-1}}$ is numerically equivalent to $(n-1)!C$, where C is a positive cycle, and the 0-cycle $X_{u_1} \cdots X_{u_n}$ is of degree $n!$; then C is a curve, A is the Jacobian variety of C, X is a canonical divisor, and C is canonically embedded in A. Also the converse is shown to be true, so that this gives a full characterization of Jacobian varieties. {Some points in the proof would need clarification, and some are obviously in need of correction (e.g., on p. 11, in order to prove that a certain mapping $f = (f_1, \cdots, f_m)$ of A into a product $J_1 \times \cdots \times J_m$ is surjective, the author says that it is enough to show that each one of the f_i is surjective, and proceeds accordingly). Probably no great effort would be needed in order to remove these blemishes; it is to be hoped that a complete and flawless proof will soon be forthcoming.} *A. Weil* (Princeton, N.J.)

7214:

Igusa, Jun-ichi. Kroneckerian model of fields of elliptic modular functions. Amer. J. Math. **81** (1959), 561–577.

A continuation of the author's recent work on elliptic modular functions [Amer. J. Math. **81** (1959), 436–452; MR **21** #3421]. The main result states that there exists a nonsingular projective curve defined over the rationals that is a model for the field of modular functions of level n in characteristic zero whose reduction modulo any prime p not dividing n is a nonsingular model for the modular functions of level n in characteristic p. There follow several highly technical results in which key expressions are modular correspondence and Petersson conjecture on Hecke operators on cusp forms.

M. Rosenlicht (Berkeley, Calif.)

LINEAR ALGEBRA

See also 7437, 7440, 7478.

7215:

Dulmage, A. L.; and Mendelsohn, N. S. A note on the stochastic rank of a bipartite graph. Canad. Math. Bull. **2** (1959), 159–162.

Theorem 6 of an earlier paper by the same authors [Canad. J. Math. **11** (1959), 269–79; MR **21** #3433] relating term rank and stochastic rank is given graphical interpretation and proof. *S. Sherman* (Philadelphia, Pa.)

7216:

Taussky, Olga; and Zassenhaus, Hans. On the similarity transformation between a matrix and its transpose. Pacific J. Math. **9** (1959), 893–896.

It is well known that every matrix A is similar to its transpose. The authors prove that the transforming matrix can always be chosen symmetric; and that every transforming matrix is symmetric if and only if A is non-derogatory. *H. K. Farahat* (Sheffield)

7217:

Ostrowski, Alexander. Über Produkte Hermitescher Matrizen und Büschel Hermitescher Formen. Math. Z. **72** (1959/60), 1–15.

For an n-square matrix M, let R_M, π_M, and ν_M denote the number of real characteristic roots, the number of positive roots and the number of negative roots, respectively. Let $\bar{\pi}_M$ and $\bar{\nu}_M$ be the numbers of non-negative and non-positive roots. Let $\sigma_M = \pi_M - \nu_M$; and let $\delta_M = n - \text{rank } M$. The author shows that for any pair of Hermitian matrices A and B, $R_{AB} - \max(|\sigma_A| + \delta_A, |\sigma_B| + \delta_B)$ is an even non-negative integer. Furthermore,

$$\bar{\pi}_{AB} \geq \tfrac{1}{2}|\sigma_A + \sigma_B| + \tfrac{1}{2}(\delta_A + \delta_B),$$
$$\bar{\nu}_{AB} \geq \tfrac{1}{2}|\sigma_A - \sigma_B| + \tfrac{1}{2}(\delta_A + \delta_B).$$

If A is non-singular,

$$\pi_{AB} \geq \tfrac{1}{2}|\sigma_A + \sigma_B| - \tfrac{1}{2}\delta_B, \quad \nu_{AB} \geq \tfrac{1}{2}|\sigma_A - \sigma_B| - \tfrac{1}{2}\delta_B.$$

Other inequalities are given, relating the numbers of positive and negative roots of A and B to the numbers of non-real roots and the exponents of the elementary divisors of AB.

These results are based on the following theorem of Klein: If B and C are Hermitian and C is non-singular, and if m is the number of real elementary divisors of odd order of the determinant of $B - \lambda C$, then $m \geq |\sigma_C|$. The author gives a proof of this theorem based on the reduction theory of families of Hermitian forms, and shows that $m - |\sigma_C|$ is even. *B. N. Moyls* (Vancouver, B.C.)

7218:

Marcus, M.; Moyls, B. N.; and Westwick, R. Extremal properties of Hermitian matrices. II. Canad. J. Math. **11** (1959), 379–382.

[For part I, see M. Marcus and J. L. McGregor, same J. **8** (1956), 524–531; MR **18**, 273.] Let H be an $n \times n$ Hermitian matrix with eigenvalues $h_1 \geq \cdots \geq h_n$. Let Q_{kr} denote the $\binom{k}{r}$ sets of integers $(i_1, \cdots, i_r)$ such that $1 \leq i_1 < i_2 < \cdots < i_r \leq k$, where $1 \leq k \leq n$; for any $\omega \in Q_{kr}$ let x_ω be the Grassmann product $x_{i_1} \wedge \cdots \wedge x_{i_r}$. Let $E_r(a_1, \cdots, a_k)$ be the rth elementary symmetric function of the numbers $a_1, \cdots, a_k$ and $g = \sum_{\omega \in Q_{kr}} (C_r(H)x_\omega, x_\omega)$, where $C_r(H)$ is the rth compound of H. Then

$$\max g = \max_{0 \leq s \leq k} E_r(h_1, \cdots, h_s, h_{n-k+s+1}, \cdots, h_n),$$
$$\min g = \min_{0 \leq s \leq k} E_r(h_1, \cdots, h_s, h_{n-k+s+1}, \cdots, h_n),$$

the extrema being taken over all orthonormal sets $x_1, \cdots, x_k$ of k vectors. The corresponding result when H is positive definite was proved in part I.

H. S. A. Potter (Aberdeen)

7219:

Marcus, Marvin; and Moyls, B. N. Transformations on tensor product spaces. Pacific J. Math. **9** (1959), 1215–1221.

Let $M_{m,n}$ denote the space of all $m \times n$ matrices over an algebraically closed field of characteristic 0. The object of the paper under review is to elucidate the structure of those linear transformations of $M_{m,n}$ into itself which map matrices of rank 1 into matrices of rank 1. It is found convenient to regard $M_{m,n}$ as a tensor product of two vector spaces and to frame the argument in the language of multilinear algebra.

Related questions concerning linear transformations which preserve certain features of $M_{m,n}$ were considered earlier by the authors [Canad. J. Math. **11** (1959), 61–66; MR **20** #6432] and by M. Marcus and R. Purves [ibid. **11** (1959), 383–396; MR **21** #4167]. *L. Mirsky* (Sheffield)

7220:

Bellman, Richard. Representation theorems and inequalities for Hermitian matrices. Duke Math. J. **26** (1959), 485–490.

It is known that (1) $\int_{-\infty}^{\infty} e^{-(x,Ax)}dx = \pi^{n/2}/|A|^{1/2}$ when A is a symmetric $n \times n$ matrix whose real part is positive definite, $dx = dx_1 \cdots dx_n$ and the integration is over the whole x-space. It is shown here that (2) $\int_{-\infty}^{\infty} e^{-(z,Hz)}dxdy = \pi^n/|H|$ when $z = x + iy$ and H is a positive definite Hermitian matrix. The result (1) was used by the author previously to derive various inequalities. Using (2) the following generalization of a theorem of Hua [Acta Math. Sinica **5** (1955), 463–470; MR **17**, 703] is obtained: If H_i are complex $n \times n$ matrices such that $I - H_i^* H_i$ is positive definite hermitian for $i = 1, 2, \cdots$ then (3) the matrix $(|I - H_i^* H_j|^{-k})$ is positive definite for $k = 1, 2, \cdots$. Alternatively, it is shown that for real H_i (3) holds for any $k < (1-n)/2$. This is deduced from an identity found by Ingham [Proc. Cambridge Philos. Soc. **29** (1933), 271–276] and Siegel [Ann. of Math. (2) **36** (1935), 527–606] for real symmetric matrices and by Braun [ibid. **50** (1949), 827–855; MR **11**, 333] in the hermitian case.

O. Taussky-Todd (Pasadena, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 7257, 7270.

7221:

★Séminaire Paul Dubreil et Charles Pisot; 10e année: 1956/57. Algèbre et théorie des nombres. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. iii + 232 pp. (mimeographed)

This volume consists of the notes on 24 weekly meetings of the seminar of Dubreil and Pisot in Paris. The subjects covered are diverse, most being accounts of original research of the speaker while a few are reports of other research, particularly some recent Russian work. Seven of the seminars were given by Michel Zisman and deal with problems of cohomology and algebraic geometry. Two papers by Mme. Levy-Bruhl deal with the relation of algebraic surfaces to certain braid groups. Two papers treat recent Russian work on semi-groups. O. Zariski gives a criterion for the birational equivalence of a surface with a ruled surface. L. Lesieur discusses a problem in the foundations of geometry, giving an algebraic theory of orientation. Several papers discuss the theory of rings, in particular rings of valuations, and Noetherian rings. Mlle. Chamfy applies the theory of meromorphic functions to study certain closed sets of algebraic integers.

Marshall Hall, Jr. (Pasadena, Calif.)

7222:

★Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot; 11e année: 1957/58. Algèbre et théorie des nombres. 2 Vols. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. Vol. 1. iii + 146 pp.; Vol. 2. iii + 128 pp. (mimeographed)

These two volumes, like the one reviewed above, contain the notes of 25 weekly meetings of the seminar of Dubreil, Dubreil-Jacotin and Pisot in Paris. Most of these are again original work, but a few are reports on the work of others. Four meetings were conducted by Michel Zisman but the notes for two of these were combined. These deal with the cohomology of algebraic varieties. Julien Petresco gives a new proof of the theorem of Kurosh on the subgroups of free products. M. Schutzenberger extends the Schreier theorem on subgroups of free groups to free semi-groups. In two papers Guy Maury discusses commutative rings, in one quoting the work of Northcott on local rings, in another giving some transference theorems from a ring to a simple algebraic extension. Four papers discuss Noetherian rings, two by P. Jaffard on the dimension of polynomial rings, one by L. Lesieur and another by R. Croisot. Lombardo-Radice discusses finite quasifields (also known as Veblen-Wedderburn systems). Mme. F. Bertrandais treats entire functions taking on integral values for positive integral arguments. H. Delange studies densities of sets of integers, in particular the square-free numbers in an arithmetical progression. J. Benabou treats local lattices, and E. A. Behrens discusses rings represented by modules with a distributive lattice of submodules.

Marshall Hall, Jr. (Pasadena, Calif.)

7223:

Mori, Yoshiro. On the integral closure of an integral domain. VI. On the notion of Artin's symbols. II. Bull. Kyoto Gakugei Univ. Ser. B **14** (1959), 1–3.

Continuation of the study of F-ideals in an integral domain R [see the preceding part, same Bull. **13** (1958), 1–3; MR **21** #3447]. They form a modular lattice. Relations with quasi-equality. *P. Samuel* (Urbana, Ill.)

7224:

★Laman, Gerard. On automorphisms of transformationgroups of polynomial algebras. Thesis, Rijksuniversiteit te Leiden, 1959. 77 pp.

Let A be the ring of polynomials in n indeterminates without constant term over a field k of characteristic zero. Let A_s be the quotient algebra obtained by reducing A modulo the ideal generated by the polynomials of degree $\geq s+1$. The author studies the structure of the

automorphism groups L_s of the algebras A_s. The factor groups and quotient groups are given explicitly. A precise description of the results would involve too many technicalities to be given in this review. The main tools used are those of multilinear algebra. The author also studies the automorphisms of the groups L_s and shows that, except for two exceptional cases which are analyzed in detail, every such automorphism arises from an automorphism of the field k and an inner automorphism of L_s. The motivation for this investigation lies in certain questions in differential geometry. The groups L_s had been studied earlier, from a local point of view, by A. Nijenhuis [Thesis, Amsterdam, 1952]. From a purely algebraic point of view, it perhaps is useful to remark that the author's results give the structure of the automorphism groups of a large class of commutative nilpotent algebras.

W. E. Jenner (Lewisburg, Pa.)

7225:

★Nöbauer, Wilfried. Die Operation des Einsetzens bei Polynomen in mehreren Unbestimmten. J. Reine Angew. Math. **201** (1959), 207–220.

The present paper generalizes the results of an earlier paper [Math. Ann. **134** (1958), 248–259; MR **20** #4549]. Let $\mathfrak{r}$ be a commutative ring with identity ϵ. Put $\mathfrak{x}=(x_1, \cdots, x_k)$, where $x_1, \cdots, x_k$ are indeterminates, and let $\mathfrak{R}=\mathfrak{r}(\mathfrak{x})$. Let R be the direct sum of k copies of $\mathfrak{R}$; the elements of R are called polynomial vectors over $\mathfrak{r}$: $\mathfrak{f}(\mathfrak{x})=(f_1(\mathfrak{x}), \cdots, f_k(\mathfrak{x}))$. Then in R, in addition to the ring operations, there is also the associative operation $\circ$ defined by

$$\mathfrak{f}(\mathfrak{x}) \circ \mathfrak{g}(\mathfrak{x}) = \mathfrak{f}(\mathfrak{g}(\mathfrak{x})) = (f_1(\mathfrak{g}(\mathfrak{x})), \cdots, f_k(\mathfrak{g}(\mathfrak{x}))).$$

The author calls R an algebra of type $\langle 2, 2, 2 \rangle$.

Among the results obtained in the first part of the paper are the following. (1) The congruence relations of the algebra R coincide with the congruence relations of the ring R modulo ideals A that satisfy $\mathfrak{f}_1(\mathfrak{x}) \equiv \mathfrak{g}_1(\mathfrak{x})$, $\mathfrak{f}_2(\mathfrak{x}) \equiv \mathfrak{g}_2(\mathfrak{x}) \pmod{A} \Rightarrow \mathfrak{f}_1(\mathfrak{f}_2(\mathfrak{x})) \equiv \mathfrak{g}_1(\mathfrak{g}_2(\mathfrak{x})) \pmod{A}$. Such ideals are called Vektorvollideale. (2) The ideal $A \subseteq R$ is a Vektorvollideal provided it is the direct sum of k copies of a Vollideal $\mathfrak{A} \subseteq \mathfrak{R}$. (3) To every ideal $\mathfrak{A} \subseteq \mathfrak{R}$ corresponds a unique ideal $\mathfrak{a} \subseteq \mathfrak{r}$ such that $(\mathfrak{a}) \subseteq \mathfrak{A} \subseteq \{\mathfrak{a}\}$. Here $(\mathfrak{a})$ is the ideal in $\mathfrak{R}$ generated by $\mathfrak{a}$, while $\{\mathfrak{a}\}$ is the set of all polynomials $w(\mathfrak{x}) \in \mathfrak{R}$ such that $w(\mathfrak{t}) \equiv 0 \pmod{\mathfrak{a}}$ for every vector $\mathfrak{t}$ over $\mathfrak{r}$; $\{\mathfrak{a}\}$ is a Vollideal. (4) The correspondence of the last theorem yields a homomorphic mapping of the algebra of the Vollideale of $\mathfrak{R}$ onto the ideal algebra of $\mathfrak{r}$. (5) The set of all Vektorvollideale of R constitutes a subalgebra of the ideal algebra of R that is isomorphic to the algebra of the Vollideale of $\mathfrak{R}$.

The results in the remainder of the paper are too complicated for brief statement.

L. Carlitz (Durham, N.C.)

7226:

Cashwell, E. D.; and Everett, C. J. The ring of number-theoretic functions. Pacific J. Math. **9** (1959), 975–985.

Let F_ω be the ring of all formal power series in a countably infinite number of indeterminates $x_1, x_2, x_3, \cdots$. The authors prove that the theorem of unique factorization into primes is true in F_ω. In the proof they make use of the fact that (for $l=1, 2, \cdots$) unique factorization holds in the ring F_l of all power series in l indeterminates $x_1, \cdots, x_l$ [W. Rückert, Math. Ann. **107** (1932), 259–281; W. Krull, Math. Z. **42** (1937), 745–766]. An important lemma: If the series $P(x_1, x_2, \cdots)$ is a prime of F_ω, then for all sufficiently large values of l the series

$$P(x_1, \cdots, x_l, 0, 0, \cdots)$$

is a prime in F_l.

It is well known that F_l is isomorphic to the ring of all formal ordinary Dirichlet series, or, what amounts to the same thing, the ring of all number-theoretic functions (multiplication defined by the usual composition). Accordingly, part of the paper has been written in the language of number-theoretic functions.

N. G. de Bruijn (Amsterdam)

7227:

Nagata, Masayoshi. Note on a chain condition for prime ideals. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 85–90.

Let R, R' be two local domains such that R' dominates R; K, K' their quotient fields; $\mathfrak{m}$, $\mathfrak{m}'$ their maximal ideals; denote by d [resp. s] the transcendence degree of K' over K [resp. of $R'/\mathfrak{m}'$ over $R/\mathfrak{m}$]. We say that the dimension formula holds for R if $\dim(R)+s=\dim(R')+d$ whenever R' is a ring of quotients of a finitely generated ring extension of R. We say that the second chain condition holds in R if, in every integral extension [or, equivalently, every finite integral extension, or the integral closure] S of R, every maximal chain of prime ideals has length $\dim(S)$. The author proves that, if the second chain condition holds for a noetherian domain I, then the dimension formula holds for every local domain R which is a ring of quotients of a finitely generated ring extension of I. The author states an open problem, related with the question as to whether the ideal (0) of the completion of a local domain may have imbedded prime ideals.

P. Samuel (Urbana, Ill.)

7228:

Kertész, Andor. Investigations in the theory of operator modules. III. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 105–120. (Hungarian)

See the review of parts I and II [same J. **8** (1958), 411–436, **9** (1959), 15–50; MR **21** #3453]. In an appendix a radical $\mathfrak{R}(M)$ is defined for operator modules M over a ring R as the set of all $x \in M$ such that Rx belongs to all maximal submodules of M; if no maximal submodules exist, then $\mathfrak{R}(M)=M$. This $\mathfrak{R}(M)$ corresponds to and has similar properties to the Jacobson radical of a ring.

L. Fuchs (Budapest)

7229:

Reiner, Irving. On the class number of representations of an order. Canad. J. Math. **11** (1959), 660–672.

Let R be a Dedekind ring with the field of quotients K. Let p be, in general, a prime ideal of R, and let R_p be the ring of p-adic integers in K. Assume that, for every p, R/p consists of a finite number of elements and that the class number h of R is finite.

Let A be a finite-dimensional separable algebra over K. K is assumed to be a splitting field for A. Let G be an R-order of A, i.e., G is a subring of A satisfying (i) $1 \in G$, (ii) G contains a K-base of A and (iii) G is a finitely generated R-module. By a G-module, we mean a left G-module which is a finitely generated torsion-free R-module,

on which the identity of G acts as identity operator. A-modules are defined similarly.

$A_p = R_p \otimes_R G$ is an algebra over R_p. For a G-module M, we denote by KM and M_p the A-module $K \otimes_R M$ and the A_p-module $R_p \otimes_R M$. Two G-modules M and N are said to be in the same genus (notation: $M \mathbf{v} N$) if the modules M_p and N_p are A_p-isomorphic for every p. It is known that $M \mathbf{v} N$ if and only if $KM \cong KN$ and $M_p \cong N_p$ for every p.

For an A-module L', let $S(L')$ be the collection of G-modules L for which $KL \cong L'$. Suppose that $S(L')$ splits into r_g genera, and into r_G classes under G-isomorphism. Maranda [same J. **7** (1955), 516–526; MR **19**, 529] proved that, if L' is irreducible, then $r_G = hr_g$.

The present article proves that $r_G \geq h^k r_g$ if L' has k distinct irreducible constituents. A sufficient condition for the validity of the equality is given, and, in the special case where $k=2$, some formulas for r_g and r_G are given which show that r_G can be different from $h^k r_g$.

M. Nagata (Kyoto)

7230:

Slowikowski, W. On regular embeddings of a group with differential operators. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 119–124. (Russian summary, unbound insert)

Let R be a ring with 1. The author generalizes the notion of module over R, thereby defining that of "d-group", by permitting each element $A \in R$ to operate on only some of the elements of an additive group X, these elements forming a subgroup G_A of X; furthermore, the module conditions $(A+B)x = Ax + Bx$ and $(AB)x = A(Bx)$ are no longer assumed. If $G_A = X$ for every A the d-group is called closed. Consider a homomorphism $h: X \to Y$ of d-groups, in the obvious sense. h is "almost onto" if, whenever Z is a closed d-group and $k: X \to Z$ and $g: Z \to Y$ are homomorphisms with $gk = h$, then g is onto Y; h is a "regular embedding" if Y is a module. In the latter case the submodule M_h of $R \otimes X$ consisting of all $\sum A_i \otimes x_i$ with $\sum A_i h x_i = 0$ is called the quasi-kernel of h. Sample theorem: Let $h_i: X \to Y_i$ be a regular embedding ($i = 1, 2$) with h_1 almost onto; in order that $M_{h_1} \subset M_{h_2}$ it is necessary and sufficient that there exist a homomorphism $k: Y_1 \to Y_2$ such that $h_2 = kh_1$; k is unique and is onto if and only if h is almost onto. {A number of errors, some typographical, inhibit the reading of the paper.}

E. R. Kolchin (New York, N.Y.)

7231:

Slowikowski, W. Categories and representations of groups with differential operators. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 125–130. (Russian summary, unbound insert)

This note continues the one reviewed above. Let X be a fixed d-group over a ring R. In the first part the author considers a class of regular embeddings of X into R-modules belonging to a category, the class and category being subject to a certain compatibility condition. For example, if X is a topological group then the category can be that of topological R-modules and the class can be that of the continuous regular embeddings. In the second part he considers, for given additive group U and given R-module Φ of balanced bi-additive mappings of $R \times X$ into U, the module X_Φ of group homomorphisms $\Phi \to U$ generated by the group homomorphisms of the form $\varphi \to \varphi(A, x)$. The mapping $h_\Phi: X \to X_\Phi$ such that $h_\Phi(x)$ is the group homomorphism $\varphi \to \varphi(1, x)$ is then a regular embedding almost onto X_Φ, and the quasi-kernel M_h is easily described. This result is asserted to be analogous to an existence theorem for distributions due to L. Schwartz.

E. R. Kolchin (New York, N.Y.)

7232:

Kertész, Andor; and Steinfeld, Otto. Characterizations of semi-simple rings. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 301–314. (Hungarian)

Ten different characterizations of semi-simple rings R are given, including the classical definition, Noether's decomposition and the Wedderburn-Artin structure theorem; the others are concerned with ideals of R. Two typical ones are: R has zero radical and is the direct sum of a finite number of minimal quasi-ideals (i.e., submodules Q with $QR \cap RQ \subseteq Q$); R has a right unit element and some finite set of maximal left ideals has 0 intersection. The proof is cyclical.

L. Fuchs (Budapest)

7233:

Walter, John H. Structure of cleft rings. I. Illinois J. Math. **3** (1959), 445–467.

A ring R with radical N and identity element 1 is called cleft in case R satisfies the minimum condition and $R = S \oplus N$ as an additive group, where $S \cong R/N$ is semi-simple. The author finds a condition that for two cleft rings $R = S \oplus N$ and $R' = S' \oplus N'$ an isomorphism [anti-isomorphism] of S onto S' be extendable to an isomorphism [anti-isomorphism] of R onto R'. When both R and R' are finite-dimensional algebras over a field K, the condition for extendability of an anti-isomorphism is rephrased so as to provide a condition that certain finite-dimensional cleft algebras be commutative.

To formulate these results for cleft rings R and R', the author introduces the notion of a "structure" of a left R-module X. Let S_i, F_i ($i = 1, \cdots, k$) be complete lists of the simple ideals of S and of the non-isomorphic irreducible left R-modules, respectively. Let K_i be the endomorphism sfield of F_i ($i = 1, \cdots, k$). The author shows that no generality is lost in assuming that R is an algebra over a field K. For $f_i \in \mathrm{Hom}_S(F_i, X)$, $g_j \in \mathrm{Hom}_S(X, F_j)$, and $\alpha \in R$, $\alpha \to g_j \alpha_L f_i$ defines an element $\psi_{ji}[g_j, f_i]$ of $H_{ji} = \mathrm{Hom}_{(S,\,S)}(R, \mathrm{Hom}_K(F_i, F_j))$ ($i, j = 1, \cdots, k$). Then the set of mappings ψ_{ji} ($i, j = 1, \cdots, k$) is called the structure $|\psi|$ of X. The first theorem asserts that the isomorphism of two left R-modules X, X' is equivalent to the existence of K_i-isomorphisms φ_i of $\mathrm{Hom}_S(F_i, X)$ onto $\mathrm{Hom}_S(F_i, X')$ and ρ_i of $\mathrm{Hom}_S(X, F_i)$ onto $\mathrm{Hom}_S(X', F_i)$ such that $(\rho_i g_i)(\varphi_i f_i) = g_i f_i$ and $\psi_{ji}[g_j, f_i] = \psi_{ji'}[\rho_j g_j, \varphi_i f_i]$ for $i, j = 1, \cdots, k$. The author then studies the (K_j, K_i)-modules $M(T) = \mathrm{Hom}_{(S_j,\,S_i)}(T, \mathrm{Hom}_K(F_i, F_j))$ for a given (S_j, S_i)-module T. When S_i and S_j are replaced by isomorphic copies, then isomorphisms of $M(T)$ induce isomorphisms of T (and vice-versa) and these isomorphisms are related in a natural manner stated in theorem 2. The author now considers the principal indecomposable left R-modules U_q, labeled so that U_q/NU_q is isomorphic to F_q ($q = 1, \cdots, k$). He combines his theorems 1 and 2 to derive his main result. With $R = S \oplus N$ and $R' = S' \oplus N'$ cleft rings, let I_θ be an isomorphism of S onto S' (which maps S_i onto S_i') so that there is an I_θ-isomorphism ω_i

of F_i onto F_i' $(i=1, \cdots, k)$. Let $\psi_{ji,q}$ $(i, j=1, \cdots, k)$ be the structure of U_q $(q=1, \cdots, k)$. Then I_0 can be extended to an isomorphism of R onto R' if and only if there are I_0-isomorphisms φ_{jq} of $\mathrm{Hom}_S(F_i, U_q)$ onto $\mathrm{Hom}_{S'}(F_i', U_q')$ and ρ_{iq} of $\mathrm{Hom}_S(U_q, F_i)$ onto $\mathrm{Hom}_{S'}(U_q', F_i')$, and (I_0, I_0)-isomorphisms θ_{ji} of H_{ji} onto H_{ji}' such that

$$(\rho_{iq}g_{iq})(\varphi_{iq}f_{iq}) = \omega_i g_{iq} f_{iq} \omega_i^{-1}$$

and

$$\theta_{ji}\psi_{ji,q}[g_{jq}, f_{iq}] = \psi_{ji,q}'[\rho_{iq}g_{jq}, \varphi_{iq}f_{iq}]$$

for all

$$g_{jq} \in \mathrm{Hom}_S(U_q, F_j), \quad f_{iq} \in \mathrm{Hom}_S(F_i, U_q)$$

$(i, j, q=1, \cdots, k)$. The condition for extendability of an anti-isomorphism is analogous. If R is finite-dimensional over K, and S is in the center of R, then R is commutative if and only if $H_{ji}=0$ for $j \neq i$ and the structures $|\psi_q|$ of U_q and $|\zeta_q|$ of U_q^* are related in a manner analogous to that required of $|\psi_q|$ and $|\psi_q'|$ in the author's main result.

The proofs make essential use of only a bare minimum of homological algebra. The key lemmas concern a necessary and sufficient condition in terms of the modules $\mathrm{Hom}_S(F_i, X)$ and $\mathrm{Hom}_S(X, F_i)$ for representing a left R-module X as an S-direct sum, and a sufficient condition that $RfF_i = U_i$ for $f \in \mathrm{Hom}_S(F_i, U_i)$. The remaining arguments are typical of discussions concerning linear algebra. *M. F. Smiley* (Iowa City, Iowa)

7234:

Vinograde, B.; and Weeg, G. P. Maximally uncleft rings and algebras. Illinois J. Math. **3** (1959), 272–284.

Let A be a finite-dimensional algebra over a field F or a semi-primary ring with nilpotent radical N. A is called cleft if there exists a subring A^* such that A^* is semisimple and $A^*+N=A$ holds; otherwise it is called uncleft. A is called maximally uncleft (briefly, m.u.) if A/J is uncleft for every ideal J in the radical $(N \neq 0, J \neq N)$. An algebra A such that A/N is separable is cleft (Wedderburn-Malcev) and a complete equal characteristic local ring is cleft (I. S. Cohen). But $C(p^r)$ (C: ring of integers), $F(\alpha)/F$ (F is characteristic p, α satisfies minimum equation $(x^q-c)^n=0$, $q=p^e$, $n>1$, x^q-c is irreducible in F), and a commutative unequal characteristic complete primary ring are simple examples of m.u. rings. This paper consists of three parts. In part I some elementary properties are considered. A ring is m.u. if and only if it is the sum of primary rings at least one of which is m.u. while the others are simple if not m.u. A primary ring B_n is m.u. if and only if the complete primary ring B is m.u. A commutative primary ring A is m.u. if and only if A is of unequal characteristic and a coefficient ring [cf. A. Geddes, J. London Math. Soc. **29** (1954), 334–341; MR **16**, 213]. In part II a m.u. ring is characterized by means of cohomology of a ring. An algebra A is m.u. if and only if A/N^2 is m.u. Let A be a completely primary ring such that N is contained in the center of A. A/N^2 is characterized by the 2-cocycle $g(x, y)=A_xA_y-A_{xy}$ of the ring $R=A/N$ over the R-R-module N/N^2, where x, y are elements of R and A_x is a representative of x in A/N^2. Let $n_1, \cdots, n_d$ be a left basis of N over R and let $g(x, y)=\sum p_i(x, y)n_i$. Then A is m.u. if and only if $p_1, \cdots, p_d$ are a linearly independent (over R) set of non-cobounding cocycles of R over R. In part III it is proved that the product of two division algebras over F is m.u. if and only if the product of their centers is m.u. *Y. Kawada* (Tokyo)

7235:

Szász, Ferenc A. Die explizite Bestimmung von einigen Klassen der assoziativen Ringe. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 107–110. (Russian summary, unbound insert)

The author considers four properties which may hold for an associative ring R. These properties are: E_1: if R_1 and R_2 are finitely generated subrings of R such that $0 \neq R_i \neq R$, for $i=1, 2$, then $R_1 \simeq R_2$; E_2: if R_1 is a subring of R, $R_1 \neq R$, then there exists $r \in R$ such that $R_1 = R \cdot r$; E_3: if R_1 is a subring of R, $R_1 \neq R$, then there exists $r \in R$ such that $R_1 = R \cdot r + I \cdot r$, where I is the ring of integers; E_4: if L is a left ideal of R, then there exist a nil left ideal L_1 and an idempotent e of R such that $L = R \cdot e + L_1$. For each of the properties E_1, E_2, and E_3, the author gives, without proof, an explicit list of all associative rings R having that property. The author states, also without proof, that a ring R has property E_4 if and only if R has a two-sided nil ideal N such that R/N is a semisimple ring with minimum condition for left ideals. *Drury W. Wall* (Iowa City, Iowa)

7236:

Nakayama, Tadasi; and Tsuzuku, Tosiro. A remark on Frobenius extensions and endomorphism rings. Nagoya Math. J. **15** (1959), 9–16.

Let $\mathfrak{S}$ be a ring with unit element 1 and let A be a subring of $\mathfrak{S}$ such that $1 \in A$. The author calls $\mathfrak{S}$ a Frobenius extension of A if $\mathfrak{S}$ has a finite independent A-right basis and if there exists an A-$\mathfrak{S}$-isomorphism between the A-$\mathfrak{S}$-modules $\mathfrak{S}$ and $\mathrm{Hom}_A(\mathfrak{S}_A, \mathfrak{S}_A)$. If A is an S-ring then this definition agrees with that given by Kasch in his paper [Math. Ann. **127** (1954), 453–474; MR **16**, 7] in which the theory of Frobenius extensions was first introduced. (A ring is an S-ring if it is Artinian and the left annihilator of each proper right ideal as well as the right annihilator of each proper left ideal is not zero.) The purpose of the present paper is to free one of Kasch's theorems [loc. cit., Satz 5] from the S-ring hypothesis. Let $\mathfrak{E}=\mathrm{Hom}_A({}_A\mathfrak{S}, {}_A\mathfrak{S})$ and let $\mathfrak{S}_r=\{\phi : \mathfrak{S} \to \mathfrak{S} \mid$ for some s and all $x \in \mathfrak{S}$, $\phi(x)=xs\}$. Theorem: (A) If $\mathfrak{S}$ is a Frobenius extension of A, then $\mathfrak{E}$ is a Frobenius extension of $\mathfrak{S}_r$. (B) If $\mathfrak{E}$ is a Frobenius extension of $\mathfrak{S}_r$ and $\mathfrak{S}$ has a finite independent A-left basis, then $\mathfrak{S}$ is a Frobenius extension of A. Kasch's proof carries over for part (A) with slight modifications. The proof of (B) uses the left-right symmetry of the notion of Frobenius extension as well as the following characterization. If $\mathfrak{S}$ has a finite independent A-right basis, then $\mathfrak{S}$ is a Frobenius extension of A if and only if there exists an A-A-homomorphism $\Lambda : \mathfrak{S} \to A$ such that (i_r) if $s \in \mathfrak{S}$ and $\Lambda(s\mathfrak{S})=0$, then $s=0$; (i_l) if $s \in \mathfrak{S}$ and $\Lambda(\mathfrak{S}s)=0$, then $s=0$; (ii) $\mathrm{Hom}_A(\mathfrak{S}_A, A_A)=\Lambda\mathfrak{S}$. *Drury W. Wall* (Iowa City, Iowa)

7237:

Faith, Carl C. Submodules of rings. Proc. Amer. Math. Soc. **10** (1959), 596–606.

Let A be a simple algebra over a field Φ, containing a non-trivial idempotent. It is known that these algebras do not contain proper invariant subalgebras B which are not contained in the center [S. A. Amitsur, same Proc. **7** (1956), 987–989; MR **18**, 557; and for local matrix rings, A. Rosenberg, ibid. **7** (1956), 891–898; MR **18**, 462].

The author deals with the question of the number of

conjugates of these subalgebras B. He proves that if A is not the 2×2 matrix algebra over a field of characteristic 2, then there is an element $u\in A$, $u^2=0$, such that the cardinal number of the set $\{(1+\alpha u)^{-1}B(1+\alpha u), \alpha\in\Phi\}$ = cardinal number of Φ, if the latter is infinite. This follows as a consequence of a more general result about submodules of simple rings containing a non-trivial idempotent. For local matrix rings A of degree ≥ 3 over simple rings, it is shown that any proper subring B which is not contained in the center has infinitely many conjugates of the form $(1+q)^{-1}B(1+q)$. The paper concludes with some structure results of these simple algebras A over fields $\Phi\neq GF(2)$: (1) A is a sum of isomorphic principal right ideals with idempotent generators; (2) every element of A is a sum of quasi regular elements; (3) A is not a radical extension of any proper subalgebra.

S. A. Amitsur (New Haven, Conn.)

7238:

De Cicco, John. Introduction to the theory of a quadratic extension Γ of a field K. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 223–251.

Elementary algebraic properties of an algebra Γ/K with unit and of rank 2 over a field K are treated. There are three cases: (i) Γ is a field, (ii) Γ is semi-simple and (iii) Γ has a radical. The author calls Γ/K elliptic, hyperbolic, parabolic, respectively, in the above three cases. He considers some formal analogues of the angle and the length in the Gaussian plane over K in cases (i) and (ii).

Y. Kawada (Tokyo)

7239:

Steinfeld, O. Über die Strukturs̈ätze der Semiringe. Acta Math. Acad. Sci. Hungar. **10** (1959), 149–155. (Russian summary, unbound insert)

Let S be a semiring (with commutative addition), containing 0 ($0S=S0=0$) and 1, which is a strong (=unique expressibility) direct sum of minimal left ideals. Then S can be written as $S=\sum Se_i$, $e_i^2=e_i$, $e_ie_j=0$, $i\neq j$, where the $\{Se_i\}$ are minimal left ideals, and Se_i is left S-module isomorphic to Se_j if and only if $e_iSe_j\neq 0$. Then (*) if $0\neq a_{ij}\in e_iSe_j$, then there exists $a_{ji}^*\in e_jSe_i$ satisfying $a_{ji}^*a_{ij}=e_j$, $a_{ij}a_{ji}^*=e_i$. The sum S_1 of those e_iSe_j which are S-isomorphic to Se_1 is a two-sided ideal of S. If $S_1=Se_1+\cdots+Se_h$, and if $d_i^*\in e_1Se_i$ and $d_i\in e_iSe_1$ {chosen by (*)} satisfy $d_i^*d_i=e_1$, $d_id_i^*=e_i$, $i=1,\cdots,h$, then the mapping $a\to(d_i^*ad_j)$ of S_1 into the semiring D_h of all $h\times h$ matrices with elements $\{d_i^*ad_j\}$ in the division semiring $D=e_1Se_1$ is an isomorphism. Furthermore, S is a strong direct sum of such semirings. In this way (completely analogous to the author's development [in Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 63–65; MR **21** #3457] of the Wedderburn-Artin structure theorem for rings) the author establishes a result which Bourne and Zassenhaus earlier proved [in Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 613–615; MR **19**, 726] under the following additional assumptions: (i) S contains no nilpotent one-sided ideals, and (ii) every two-sided ideal of S contains both a minimal left and a minimal right ideal of S. Consequently, the semirings considered by the author satisfy (i) and (ii). The author proceeds to show that these semirings are also strong direct sums of minimal right ideals. All of this is done in an independent, self-contained way. In fact, the author's proof for semirings is shorter than the complete proof for rings given by him earlier, since the initial conditions are so much stronger. This emphasizes that the following problem is still open: Determine suitable conditions in order that a semiring S be a strong direct sum of minimal left ideals.

Carl Faith (Heidelberg)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

7240:

Albert, A. A.; and Paige, L. J. On a homomorphism property of certain Jordan algebras. Trans. Amer. Math. Soc. **93** (1959), 20–29.

Theorem: Let $\mathfrak{D}$ be any linear algebra (associative or not) over a field of characteristic $\neq 2$, with an involution T. If the algebra $\mathfrak{H}_0$ of all 3-rowed T-Hermitian matrices over $\mathfrak{D}$ relative to Jordan-multiplication is a homomorphic image of a special Jordan algebra, then $\mathfrak{D}$ is associative.—The authors establish this theorem by showing how to reconstruct the multiplication in $\mathfrak{D}$ from the Jordan-multiplication in $\mathfrak{H}_0$. The result implies in particular that the exceptional simple Jordan algebra $\mathfrak{H}$ of dimension 27 is not a homomorphic image of a special Jordan algebra. Since $\mathfrak{H}$ can be generated by 3 elements, it follows that special Jordan algebras must satisfy an identity in 3 variables which is not a consequence of the defining identities of abstract Jordan algebras. {From the authors' proof it appears that there are such identities of degree at most 45, and by a result of I. G. Macdonald [Proc. London Math. Soc. (3) **10** (1960), 395–408] they must be of degree at least two in each of the three variables.}

P. M. Cohn (Manchester)

7241:

Davis, Robert L. Torsion in Engel modules. Proc. Amer. Math. Soc. **10** (1959), 679–685.

In a Lie ring with no torsion the Engel identity

$$[\cdots[[x,y],y],\cdots,y]=0,$$

with $m-1$ y's, implies the identities $S_\mu=0$ where, for $\mu=(\mu_1,\mu_2,\cdots,\mu_r)$ and $\sum\mu_i=m-1$, S_μ denotes the homogeneous Lie polynomial obtained by putting $y=y_1+y_2+\cdots+y_r$ in $[\cdots[[x,y],y],\cdots,y]$ and picking out all terms of degree μ_i in y_i for all i. The Engel module $E(m,q)$ studied in this paper is defined by imposing all the identities $S_\mu=0$ on the free Lie ring on q generators (with integer coefficients). The additive structure of the homogeneous part of $E(m,q)$ of degree $m-2$ in a given generator and degree 1 in each of two other given generators, is determined completely. In particular, if m is a prime power p^s, and $q\geq 3$, then $E(m,q)$ contains elements of additive order p.

P. J. Higgins (London)

HOMOLOGICAL ALGEBRA

See also 7184, 7233.

7242:

Hirata, Kazuhiko. On relative homological algebra of Frobenius extensions. Nagoya Math. J. **15** (1959), 17–28.

Let R be a ring which is a finitely generated free module on n generators both as a left and right module over a

subring S. If $\mathrm{Hom}_S(R, S)$ (R, S taken as right S-modules) is isomorphic to R as a left S-, right R-module, the pair (R, S) is called a Frobenius extension [Kasch, Math. Ann. **127** (1954), 453–474; MR **16**, 7]. The author begins by noting that several results on the absolute homological dimension of Frobenius algebras in S. Eilenberg and T. Nakayama, Nagoya Math. J. **9** (1955), 1–16 [MR **17**, 453] carry over to this more general situation, e.g., for any left R-module of finite R-dimension the S- and R-dimensions coincide. He is mainly concerned, however, with obtaining results for the various relative homological dimensions in the sense of Hochschild [Trans. Amer. Math. Soc. **82** (1956), 246–249; MR **18**, 278] for Frobenius extensions. Thus he shows that the left or right relative global dimension of a Frobenius extension is 0 or ∞ and that R is a relative injective module for (R, S). Furthermore, if P is an algebra over K with Q a subalgebra such that (P, Q) is a Frobenius extension, then $\dim(P, Q) = 0$ or ∞. Suppose further that P, Q are supplemented algebras with supplementation ε; then it is shown i.a. that $H^n((P, Q), M_\varepsilon) = \mathrm{Ext}^n_{(P,Q)}(K, M)$ for M a left P-module. The author concludes by constructing a complete resolution for a Frobenius extension of algebras which includes both the complete relative resolution for groups as given by Hochschild [loc. cit.] and the complete resolution for Frobenius algebras given by Nakayama [Osaka Math. J. **9** (1957), 165–187; MR **20** #6449].

A. Rosenberg (Evanston, Ill.)

7243:

Villamayor, Orlando E. On weak dimension of algebras. Pacific J. Math. **9** (1959), 941–951.

The paper contains a characterisation of algebras of weak dimension (= w.dim) zero, and various applications: Let A be a K-algebra, K commutative, then w.dim $A = 0$ if and only if for every finite set $\{a_1, \cdots, a_n\}$ of elements of A there exists an element $e \in A \otimes_K A$ such that $a_i e = e a_i$ and the image of e under the mapping: $x \otimes y \to xy$ is 1. This leads to many consequences: (1) w.dim $A = 0$ implies dim $A = 0$ for finitely generated algebras A. For infinitely generated algebras A over fields, w.dim $A = 0$ yields only that A is locally of dim ≤ 1. (2) If A is K-free, then A is locally finite. (3) For commutative algebras over field K, w.dim $A = 0$ is equivalent to local separability, or to the regularity of $A \otimes_K F$ for every field $F \supseteq K$. (4) If K is a field and A satisfies the minimum condition for left ideals then w.dim $A = 0$ is equivalent to the fact that A is semi-simple and locally finite over K and that the center of A is locally separable over K. Next the author gives simple proofs for the following: (5) The group algebra $K[G]$ of a group G is regular if (by homological methods) and only if (algebraic method) G is locally finite and K is regular, uniquely divisible by the orders of the elements of G [M. Auslander, Proc. Amer. Math. Soc. 8 (1957), 658–664; MR **19**, 390]. (6) Let A be a K-projective algebra, w.dim $A = 0$, and B be a semi-simple K-algebra in the sense of Jacobson; then $A \otimes_K B$ is also semi-simple; and consequently, if A and B are algebras over a field K, then $J(A \otimes B) = A \otimes J(B)$, where $J(\cdot)$ denotes the Jacobson radical. This generalizes and simplifies the proof of a result of the reviewer [Bull. Res. Council Israel, Sect. F **7F** (1957/58), 1–10; MR **21** #2675]. The methods of proof are homological-algebra methods.

S. A. Amitsur (New Haven, Conn.)

GROUPS AND GENERALIZATIONS

See also 7161, 7168, 7169, 7170, 7176, 7221, 7222, 7270, 7375, 7429.

7244:

Moran, S. Associative regular operations on groups corresponding to a property $\mathscr{P}$. Proc. Amer. Math. Soc. **10** (1959), 796–799.

The author gives a general method of constructing associative operations on groups. This is a continuation of the work of Golovin [Mat. Sb. (N.S.) **27 (69)** (1950), 427–454; MR **12**, 672] and three papers of the author [Proc. London Math. Soc. (3) **6** (1956), 581–596; **8** (1958), 548–568; **9** (1959), 287–317; MR **20** #3908; **20** #7054; **21** #4178]. To each property $\mathscr{P}$, applicable to groups, the author constructs a regular product. For definition of regular product see Golovin, loc. cit. The main result of the paper is: if $\mathscr{P}$ persists homomorphically and ceases freely, then the corresponding product is associative. $\mathscr{P}$ persists homomorphically if from the fact that a group satisfies $\mathscr{P}$ it follows that every homomorphic image also satisfies $\mathscr{P}$. $\mathscr{P}$ ceases freely if for every set of groups satisfying $\mathscr{P}$, their free product does not satisfy $\mathscr{P}$.

R. R. Struik (Vancouver)

7245:

Balcerzyk, S. On factor groups of some subgroups of a complete direct sum of infinite cyclic groups. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 141–142. (Russian summary, unbound insert)

Let $\mathscr{B}$ be a Boolean σ-algebra with unit V. J. Łoś defined $S(\mathscr{B})$ as the group of all sequences $a = \langle a_n \rangle$ with $-\infty < n < +\infty$, where (1) $a_n \in \mathscr{B}$ for all n, (2) $a_n \cap a_m = 0$ if $n \neq m$, (3) the union of a_n ($-\infty < n < +\infty$) is V; the sum $c = \langle c_n \rangle$ of two sequences $a = \langle a_n \rangle$ and $b = \langle b_n \rangle$ is given by the formula $c_n = \bigcup_j (a_j \cap b_{n-j})$. If $\mathscr{B}$ is the set of all subsets of some set of power m, then $S(\mathscr{B})$ is isomorphic to the complete direct sum of m copies of infinite cyclic groups. For a finitely additive ideal $\mathscr{I}$ of $\mathscr{B}$, $S(\mathscr{I})$ denotes the subgroup of all sequences $\langle a_n \rangle$ such that the union of all a_n ($n \neq 0$) belongs to $\mathscr{I}$. If $\mathscr{B}$ is a Boolean σ-algebra, $\mathscr{I}$ a finitely additive ideal of $\mathscr{B}$ and $\mathscr{I}_\beta$ the σ-ideal generated by $\mathscr{I}$, then $S(\mathscr{I}_\beta)/S(\mathscr{I})$ is a torsion-free algebraically compact group. Two further theorems of similar kind are stated; the proofs will be published in Fundamenta Mathematicae.

L. Fuchs (Budapest)

7246:

Gladkiĭ, A. V. On the nilpotency class of a group having a δ-basis. Dokl. Akad. Nauk SSSR **125** (1959), 963–965. (Russian)

The concept of δ-basis for a group has been defined by Tartakovskiĭ [Mat. Sb. (N.S.) **25 (67)** (1949), 251–274; MR **11**, 493] in his investigations of the word problem for groups. The present paper proves the following: A group with a δ-basis for $\delta < 1/2^{n+1}$, where $n \geq 2$, cannot be nilpotent of class $< n$.

L. J. Paige (Los Angeles, Calif.)

7247:

Yacoub, K. R. On general products of two finite cyclic groups one being of order 8. Proc. Math. Phys. Soc. Egypt. No. 22 (1958), 93–99 (1959). (Arabic summary)

The author continues his study of semi-special permutations and their application to the general product of two groups where, in this case, both groups are finite cyclic and one is of order eight. [See his papers, Proc. Glasgow Math. Assoc. **2** (1955), 116–123; Proc. Math. Phys. Soc. Egypt No. **21** (1957), 119–126; MR **17**, 11; **21** #4970]. He exhibits defining relations on two generators for the products corresponding to the semi-special permutations on eight symbols. However, isomorphic constructions are distinguished in only two of the six possible cases, and proof is omitted even in these cases.

F. Haimo (St. Louis, Mo.)

7248:

Hughes, D. R.; and Thompson, J. G. The H-problem and the structure of H-groups. Pacific J. Math. **9** (1959), 1097–1101.

Let G be a group, and for the fixed prime p, let $H = H_p(G)$ be the subgroup generated by those elements of G which are not of order p. It is proved that if G is finite and not a p-group, and H is neither 1 nor G, then H is of index p in G. Moreover, under the same hypothesis, H (and therefore also G) has a normal subgroup K which is nilpotent and of order prime to p, whose factor group is a p-group.

Graham Higman (Oxford)

7249:

Michler, Lothar. Über eine Teilbarkeitsbeziehung für endliche Gruppen. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **2** (1958), 5–6.

If G is a finite group of order g with a subgroup H of order h, then

$$\frac{g}{m} \Big| \left(\frac{n}{h}\right)^{g/n} \left(\frac{g}{n}\right)!,$$

where n is the order of the normalizer of H in G and m is the order of a maximal subgroup of H that is normal in G. The author deduces this fact from his results about the Brandt groupoid of algebraic extensions [#7199 above]. He also gives an independent group-theoretical proof which is due to H.- J. Hoehnke.

W. Ledermann (Manchester)

7250:

Baer, Reinhold. Kriterien für die Abgeschlossenheit endlicher Gruppen. Math. Z. **71** (1959), 325–334.

Defining r-elements g and r-groups G to be those whose orders $o(g)$ and $o(G)$ contain only primes from a given set r, a finite group G is called r-closed if the set of its r-elements forms a (characteristic) subgroup G_r. It has the Frobenius property if the order $o(G)_r$ of its set of r-elements is the greatest r-factor of $o(G)$. G is called weakly r-closed if all its subgroups U have this Frobenius property. The Frobenius property implies r-closure if G is solvable or r contains only one prime. Let r' be the set of primes not in r. Then the author shows that r-closure of G with nilpotent G/G_r is equivalent to weak r-closure of G and nilpotence of either (A) all r'-subgroups of G, or (B) all r-subgroups of G. Also that r-closure of G and solvability of G/G_r [or of $G_{r'}$] are equivalent to weak r-closure of G and the condition that the order of each simple factor group E of a subgroup of G is divisible by at most one prime from r [or from r']. The author comments that his proofs of these theorems and related lemmas make no use of group characters. Instead he makes much use of a property called r'-homogeneity—that of inducing r'-groups of automorphisms in all r'-subgroups.

J. S. Frame (East Lansing, Mich.)

7251:

Baer, Reinhold. Direkte Produkte von Gruppen teilerfremder Ordnung. Math. Z. **71** (1959), 454–457.

Let $\mathfrak{r}$, $\mathfrak{r}'$ be complementary sets of primes. A finite group is $\mathfrak{r}$-closed if its $\mathfrak{r}$-elements (elements whose orders involve only primes in $\mathfrak{r}$) form a subgroup; it is $\mathfrak{r}$-homogeneous if, for any $\mathfrak{r}$-subgroup U, the group $\mathfrak{N}U/\mathfrak{C}U$ is an $\mathfrak{r}$-group. An $\mathfrak{r}$-closed group is necessarily $\mathfrak{r}'$-homogeneous.

A finite group G is clearly the direct product of an $\mathfrak{r}$-group and an $\mathfrak{r}'$-group if, and only if, it is both $\mathfrak{r}$-closed and $\mathfrak{r}'$-closed. Such a group is therefore both $\mathfrak{r}$-homogeneous and $\mathfrak{r}'$-homogeneous. The author proves the converse of this under the following additional assumption (which is also a necessary condition for such a direct decomposition): (1) No simple factor group of a subgroup of G contains both a non-nilpotent $\mathfrak{r}$-subgroup and a non-nilpotent $\mathfrak{r}'$-subgroup. By a result of the author [#7250 above; Satz A.], this condition can be replaced by: (2) If E is a simple factor group of a subgroup of G, and if every proper subgroup of E is either an $\mathfrak{r}$-group or an $\mathfrak{r}'$-group, then E itself is either an $\mathfrak{r}$-group or an $\mathfrak{r}'$-group. It is an open question whether or not there exist simple groups E which violate (2).

P. J. Higgins (London)

7252:

Suzuki, Michio. On characterizations of linear groups. I, II. Trans. Amer. Math. Soc. **92** (1959), 191–219.

The first paper gives, generalizing results by Fowler [Thesis, Univ. of Michigan, 1951] and Brauer, Suzuki and Wall [Illinois J. Math. **2** (1958), 718–745; MR **21** #3487], a group-theoretical characterization of the 1-dimensional unimodular linear fractional group $\mathrm{LF}(2, 2^n)$ (or the 2-dimensional unimodular linear group $\mathrm{SL}(2, 2^n)$) over a finite field of characteristic 2. The main theorem states that if the centralizer of any involution in a group G of even order is abelian then (1) 2-Sylow subgroups in G are cyclic, or (2) 2-Sylow subgroups in G are normal, or (3) G is a direct product of two subgroups L and A, where $L \cong \mathrm{LF}(2, 2^n)$ with some power 2^n of 2 and A is an abelian group of odd order. The proof starts, assuming the theorem to be false, by considering a group G of smallest order for which the theorem fails to be true and showing that such G has certain properties concerning 2-Sylow groups and centralizers. It proceeds to consider generally a group which satisfies certain conditions somewhat weaker than these properties, studying its structure and characters, and deriving a formula for its order by a method similar to that of the author [Amer. J. Math. **77** (1955), 657–691; MR **17**, 580] or Brauer and Fowler [Ann. of Math. (2) **62** (1955), 565–583; MR **17**, 580]. Applied to the above G, this formula leads to a contradiction, proving the main theorem.

As an application of the main theorem it is proved that if Sylow subgroups for odd primes in a non-solvable group G are all cyclic and if two distinct 2-Sylow subgroups in G always contain only the identity in common, then G has a normal subgroup $G_0 \cong \mathrm{LF}(2, 2^n)$ such that G/G_0 is solvable and of odd order.

The second paper gives a characterization of the 2-dimensional linear fractional group over a field of characteristic 2 by the properties of involutions in it, as a counterpart to Brauer's similar characterization of the group over a field of order $\equiv -1 \bmod 4$. Let, thus, $q=2^n$ be a power of 2 and F a field with q elements. It is proved that if all involutions in a group G of even order are conjugate, and their centralizers, in G, are isomorphic to the (explicitly describable) centralizer of an involution in the 2-dimensional linear fractional group over F, then G is isomorphic to the last group provided $q>2$; in case $q=2$, G is isomorphic either with this group or with the alternating group of 6 letters. After an analysis of the structure of the centralizer of an involution, the case $q=2$ is settled rather quickly by making use of the author's former results [Illinois J. Math. **3** (1959), 255–271; MR **21** #3486]. In the long proof for the case $q>2$, both the main theorem of the first paper and its order formula are applied several times. Thus, after a complicated study of its structure and characters, G is shown to have a subgroup M of order $q^3(q-1)^2(q+1)$ and index q^2+q+1 which is the normalizer of an elementary abelian group P of order q^2. G is thus represented as a permutation group on the set $\mathfrak{P}$ of q^2+q+1 conjugates of P, and as such is doubly transitive. G contains also a subgroup L of order q^2 not conjugate to P, and a projective plane is defined where points are elements of $\mathfrak{P}$, lines are conjugates of L, and where a point lies on a line if and only if they have, as subgroups of G, a common element other than 1. It is shown to be Desarguesian, by making use of Gleason's theorem [Amer. J. Math. **78** (1956), 797–807; MR **18**, 593]. Now a short discussion on the relationship between G and the linear fractional group, both operating on the plane $\mathfrak{P}$, which makes use of the character table of the latter, brings the proof to its close.

T. Nakayama (Nagoya)

7253:

Meinhardt, Dietrich. Endliche Gruppen von Cremona-Transformationen der Ebene und die isomorphen Gruppen der zugeordneten Matrizen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 689–696.

Using the exponents of the prime divisor decompositions, admitting a finite number of divisors of the second kind relative to a fixed pair of indeterminates x, y, the author associates to a finite group of Cremona transformations of the field $k(x, y)$ relative to $k[x, y]$, k the field of all complex numbers, an isomorphic group of unimodular matrices. Furthermore, he shows how a matrix of intersection numbers of certain divisors of the second kind and at ∞, relative to z, y, may be used to prove that a finite group of Cremona transformations has a representation whose matrices leave invariant the quadratic form $-x_1^2+x_2^2+\cdots+x_n^2$, n being the number of divisors for which intersections are taken.

O. F. G. Schilling (Chicago, Ill.)

7254:

Yakubík, Yan [Jakubík, Ján]. On a class of l-groups. Časopis Pěst. Mat. **84** (1959), 150–161. (Russian. Slovak and English summaries)

The author extends the results on commutative l-groups [see Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673] to l-groups satisfying the following property: (P) If $A\subset G^+$, $x\in G^+$, there exist elements $y\in K(A)$, $z\in K'(A)$ such that $x\leqq y\cup z$, where $K'(A)$ consists of all elements $y\in G^+$ such that $x\cap y=0$ for all $x\in A$ and $K(A)=K'(K'(A))$. In particular, if G is an l-group with property (P), the author proves: If the lattice of l-ideals of G satisfies the ascending chain condition, then either G is a direct product or G contains a unique maximal proper l-ideal.

L. J. Paige (Los Angeles, Calif.)

7255:

Bauer, Heinz. Geordnete Gruppen mit Zerlegungseigenschaft. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1958**, 25–36.

Let P be a semi-group in an abelian group G with the property that $P\cap(-P)=\{0\}$. P defines an order relation in G upon setting $x\leqq y$ if and only if $y-x\in P$. A set $Q\subseteq P$ is called filtered below if $x_1, x_2\in Q$ implies the existence of an $x\in Q$ such that $x\leqq x_1$ and $x\leqq x_2$. P is called Dedekind-complete if every $Q\subseteq P$ which is filtered below has an infinum in P. Finally, P is said to possess the separation property if for every three elements a, b, $x\in P$ with $x\leqq a+b$ there exist elements $a', b'\in P$ with $a'\leqq a$, $b'\leqq b$ and $x=a'+b'$.

The author shows that if P has the separation property and is Dedekind-complete then P is a lattice. If G is a topological group, the set P is Dedekind-complete if for every $a\in P$ the set $\{x: 0\leqq x\leqq a\}$ is compact. From the previous statement it follows that if G is a topological vector space over the reals and $P\,(\subseteq G)$ a convex cone with a compact basis, then P is a lattice if and only if P possesses the separation property.

A. Devinatz (St. Louis, Mo.)

7256:

Amitsur, S. A. On the semi-simplicity of group algebras. Michigan Math. J. **6** (1959), 251–253.

Let F be a field of characteristic zero, let G be a group and let $F[G]$ be the discrete group algebra of G over F. Let Q be the field of rational numbers. The author gives, for all known cases, algebraic proofs of the semi-simplicity of $F[G]$. That $Q[G]$ contains no non-zero nil ideals is proved and then used, together with the author's earlier results [Bull. Res. Council Israel. Sect. F **7F** (1957/58), 1–10; MR **21** #2675], to show that if F is a transcendental extension of Q then $F[G]$ is semi-simple. It is proved that if G is a finitely generated commutative group then $Q[G]$ is semi-simple and that if $F[H]$ is semi-simple for each finitely generated subgroup H of G then $F[G]$ is semi-simple. Hence, if G is commutative then $Q[G]$ is semi-simple. Together with earlier results [loc. cit.] this implies that for any commutative group G and field F of characteristic zero, $F[G]$ is semi-simple. The author then proves that if every finitely generated subgroup of G has the property that all its primitive representations are of bounded degree, and if F has characteristic zero, then $F[G]$ is semi-simple. *Drury W. Wall* (Chapel Hill, N.C.)

7257:

★Green, J. A. Les polynômes de Hall et les caractères des groupes GL(n, q). Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 207–215. Centre Belge de Recherches Mathématiques. Établisse-

ments Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

An isomorphism is established between the graded algebra $\mathscr{S}$ of symmetric functions of indeterminates $x_1, x_2, \cdots$ and the "Hall algebra" of certain polynomials $P_\lambda(x, t)$ that are linear combinations of the Schur functions R_λ with integral polynomials in t as coefficients. If a_m and s_m denote the elementary symmetric functions and power sums of the x_i, then for each partition λ of n, we write $A_\lambda = \prod a_{\lambda_i}$, $S_\lambda = \prod s_{\lambda_i}$, $M_\lambda = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots$, $R_\lambda = |a_{\lambda_i+j-i}|$, thus defining four different bases for $\mathscr{S}$. Likewise each λ determines an abelian p-group G_λ of type λ that is a direct product of cyclic groups of order p^{λ_i}. If the partitions $\alpha, \beta, \cdots, \gamma$ have sum λ, the number of chains of subgroups of G_λ, each containing its successor, and having successive factor groups of types $\alpha, \beta, \cdots, \gamma$ is a polynomial in p called $g_{\alpha\beta\ldots}{}^{\lambda}{}_{\gamma}$, that is symmetric in its subscripts. The graded algebra studied by Philip Hall is based on symbols G_λ that combine by the rule $G_\alpha G_\beta = \sum g_{\alpha\beta}{}^{\lambda} G_\lambda$, and is freely generated by the symbols $G_1, G_{11}, G_{111}, \cdots$. If $\alpha, \beta, \cdots, \gamma$ are partitions of λ_1 1's, λ_2 1's, $\cdots$, and $G_\alpha G_\beta \cdots G_\gamma = \sum r_{\lambda\mu} G_\mu$, then the matrix of polynomials $r_{\lambda\mu}$ is upper triangular, with 1's on the diagonal, so it has an inverse with polynomial coefficients. Setting $P_\lambda = p^{n_\lambda} G_\lambda$ where $n_\lambda = \sum \binom{\lambda_i}{2}$, and introducing $t = 1/p$ as a new variable, we replace the polynomials $g_{\alpha\beta}{}^{\lambda}(p)$ and $r_{\lambda\mu}(p)$ by new polynomials $f_{\alpha\beta}{}^{\lambda}(t)$ and $h_{\lambda\mu}(t)$. If P_{1^n} is identified with a_n in $\mathscr{S}$, then

$$A_\lambda{}^r = \sum h_{\lambda\mu}(t) P_\mu(x, t) = \sum h_{\lambda\mu}(0) R_\mu(x).$$

The matrix $X(t)$ defined by $S_\rho(x) = \sum X_{\rho\lambda}(t) P_\lambda(x, t)$ reduces for $t = 0$ to the character table of the symmetric group of degree n. Also $P = \Phi(t) R(x)$, where Φ and $T = \Phi^{-1} = H^{-1}(t) H(0) = X(0) X^{-1}(t)$ are upper triangular matrices with integral polynomial entries. If d_ρ denotes the product of factors $(1-t^k)^{r_k}/k^{r_k}(r_k)!$ for the partition $\rho = 1^{r_1} 2^{r_2} \cdots n^{r_n}$ and $b_\rho(t)$ denotes the product over r_k of factors $(1-t) \times (1-t^2) \cdots (1-t^{r_k})$, then the author states the orthogonality relations $X'(t) \operatorname{diag}\{d_\rho\} X(t) = \operatorname{diag}\{b_\lambda(t)\}$, which he derived by group-theoretical considerations [Trans. Amer. Math. Soc. **80** (1955), 402–447; MR **17**, 345]. He suggests the desirability of a direct proof of these relations, and of finding explicit formulas for constructing P_λ.

J. S. Frame (East Lansing, Mich.)

7258:

Brauer, Richard. Zur Darstellungstheorie der Gruppen endlicher Ordnung. II. Math. Z. **72** (1959/60), 25–46.

[For part I, see Math. Z. **63** (1956), 406–444; MR **17**, 824.] The chief problem solved in this paper (theorem 7D) is that of determining the number k_τ of ordinary irreducible representations of a finite group $\mathfrak{G}$ which lie in the block B_τ, for a prime p which divides the order g of $\mathfrak{G}$. The ideas introduced in part I are further developed and lead to a refinement of the whole modular representation theory, in the sense that the block structures of certain subgroups $\mathfrak{H}$ of $\mathfrak{G}$ yield information concerning that of $\mathfrak{G}$.

In order to state the main theorem (6A), two preliminary ideas are necessary. If G is any element of $\mathfrak{G}$ it is well known that we may write $G = PR$, where P is p-singular, i.e., of order a power of p, and R is p-regular, of order prime to p; each of P and R is a power of G and $PR = RP$. If $\mathfrak{N}(P)$ is the normalizer of P in $\mathfrak{G}$, $G \in \mathfrak{N}(P)$, so that we may write

$$\chi_i(PR) = \sum_j d_{ij}{}^{P} \varphi_j{}^{P}(R),$$

where χ_i is any ordinary character of G and $\varphi_j{}^{P}$ a modular character of $\mathfrak{N}(P)$; the $d_{ij}{}^{P}$ are algebraic integers called generalized decomposition numbers, and they are independent of R. For $P = 1$, the $\varphi_j{}^{P}$ are the modular representations φ_j of $\mathfrak{G}$ and the $d_{ij}{}^{P}$ the ordinary decomposition numbers of $\mathfrak{G}$.

The second preliminary idea is that every block $\bar{B}$ of $\mathfrak{H} = \mathfrak{N}(P)$ arises by restricting $\mathfrak{G}$ to $\mathfrak{H}$ relative to some block B of $\mathfrak{G}$; this relationship is written $B = \bar{B}^{\mathfrak{G}}$, and § 2 of the paper is devoted to studying the conditions under which such a block B is uniquely defined.

Theorem 6A then reads as follows: the generalized decomposition numbers $d_{ij}{}^{P}$ can differ from zero only if χ_i belongs to the block $B = \bar{B}^{\mathfrak{G}}$. From this main theorem follow certain results concerning the characters of $\mathfrak{G}$, in particular Theorem 7D, mentioned above: If $B_1, B_2, \cdots$ B_t are the p-blocks of $\mathfrak{G}$, and if $P_1, P_2, \cdots, P_r$ is a representative set of p-singular elements of $\mathfrak{G}$ such that $l_\tau(P_i)$ is the number of irreducible modular characters $\varphi_j{}^{P_i}$ of $\mathfrak{N}(P_i)$ belonging to B with $\bar{B}^{\mathfrak{G}} = B_\tau$, then the number of ordinary irreducible characters of $\mathfrak{G}$ in B_τ is given by $k_\tau = \sum_{i=1}^{r} l_\tau(P_i)$ [cf. Masaru Osima, Canad. J. Math. **6** (1954), 511–521; MR **16**, 566].

G. de B. Robinson (Toronto, Ont.)

7259:

Ibrahim, E. M. S-functional plethysms of degrees 14 and 15. Proc. Math. Phys. Soc. Egypt. no. 22 (1958), 137–142 (2 inserts) (1959). (Arabic summary)

The reduction of $\{\lambda\} \otimes \{\mu\}$, where (λ) is a partition of n and (μ) of m, is given here for $mn = 14, 15$ [for $mn = 10, 12$, cf. same Proc. **5** (1954), no. 2, 85–86; MR **17**, 1182].

G. de B. Robinson (Toronto, Ont.)

7260:

Gluskin, L. M. Transitive semigroups of transformations. Dokl. Akad. Nauk SSSR **129** (1959), 16–18. (Russian)

Generalizations of previous work [Mat. Sb. (N.S.) **47** (**89**) (1959), 111–130; MR **21** #4197] are announced in this paper on the semigroup of transformations of a set with arbitrary structures. Several theorems, formulated in a general setting, have as special cases such results as the abstract characterization of the semigroup of all isotone transformations of a partially ordered set and the semigroup of all continuous transformations of a topological space. Similar characterizations are alluded to, but not explicitly stated, for the semigroup of all uniformly continuous transformations of a uniform topological space, the semigroup of all semilinear transformations of a linear space, the ring of all continuous linear transformations of a topological linear space and its multiplicative semigroup. A final theorem characterizes the structure of a partially ordered set by means of the semigroup of all its isotone transformations.

R. A. Good (College Park, Md.)

7261:

Ikuta, Toshiharu. On the coset decomposition of a loop. Nat. Sci. Rep. Lib. Arts Sci. Fac. Shizuoka Univ. **2** (1959), 129–133.

Several criteria are discussed that ensure the existence of a coset decomposition of a loop G relative to a subloop H. The two main conditions given can be expressed in the form (a) $y \in Hx$ implies $Hy = Hx$ for all x, y in G, and (b) $H(xy) = (Hx)y$ for all x, y in G. Both of these imply the existence of a coset decomposition mod H.

D. C. Murdoch (Vancouver, B.C.)

7262:

Whittaker, James V. On the structure of half-groups. Canad. J. Math. **11** (1959), 651–659.

The notion of a half-group was introduced by H. Furstenberg [Proc. Amer. Math. Soc. **6** (1955), 991–997; MR **17**, 1053] as a set H closed under a binary operation $x-y$ satisfying $(x-z)-(y-z)=x-y$ for all $x, y, z \in H$. As was shown by the author [Amer. Math. Monthly **62** (1955), 636–640; MR **18**, 279, 1118], a half-group containing an element e, such that $x-y=e$ if and only if $x=y$, with the operation $x+y=x-(e-y)$ becomes a group. In the present note the author defines a pseudo-group to be a half-group with an element e such that $x-x=e$. In any pseudo-group H the mapping $x \to x-e$ is a homomorphism onto a subgroup of H. If the set H is closed under a binary operation $x-y$ and has the property $(z-y)-(z-x)=x-y$, it is a pseudo-group with commutative addition $x+y=x-(e-y)$. The author then considers the left and right cancellation laws (l) $z-x=z-y$ implies $x=y$ and (r) $x-z=y-z$ implies $x=y$. He proves that a half-group in which (l) holds is a group and therefore satisfies also (r). A subset K of a half-group is called invariant if $x-(x-K) \subseteq K$, an element a is called idempotent if $a-a=a$. If the invariant sub-half-group K of a half-group H contains the set E of all idempotent elements of H, the factor half-group H/K is a group. Moreover, $F=E-E$ is an invariant sub-half-group and there exists a normal subgroup N of $G=H/F$ such that $K/F \cong N$ and $H/K \cong G/N$. The structure of half-groups which satisfy (r) is described in greater detail. Now, let H be a topological half-group such that $x \in U$ implies $x-(x-x) \in U$ for any open U. Then, if K is as above, the natural mapping from H onto H/K is an open, continuous mapping and H/K is a topological group. If, furthermore, H is connected and satisfies a certain separation axiom and if the set E of idempotents is finite, then H is a topological pseudo-group, and H is a topological group if (r) holds in H.

H. Salzmann (Frankfurt a.M.)

7263:

Stein, Sherman K. Left-distributive quasi-groups. Proc. Amer. Math. Soc. **10** (1959), 577–578.

Un quasigroupe $Q=E(\otimes)$ est distributif à gauche [Burstin-Mayer, J. Reine Angew. Math. **160** (1929), 111–130], si pour tout $a, b, c \in E$, $(a \otimes b) \otimes (a \otimes c) = a \otimes (b \otimes c)$. (i) Si $Q=E(\otimes)$ est distributif à gauche, il existe un ensemble $T \supset E$ et un groupe, défini sur cet ensemble, tel que pour tout $x, y \in E$, $x \otimes y = xyx^{-1}$. (ii) Si $G=T(\circ)$ est un groupe et si E ($E \subset T$) satisfait aux conditions que pour tout $a, b, c \in E$, $b \neq c$, b et c appartiennent à deux cosets distincts à gauche du normalisateur de a et $a \circ b \circ a^{-1} \in E$, alors $Q=E(\otimes)$, tel que $x \otimes y = x \circ y \circ x^{-1}$, est un quasigroupe distributif à gauche. (iii) Si $G=E(\circ)$ est un groupe, le groupoïde $H=E(\otimes)$ défini par $x \otimes y = x \circ y \circ x^{-1}$ est distributif à gauche [cf. M. Hosszú, Publ. Math. Debrecen **6** (1959), 1–9; MR **21** #3504].

A. Sade (Marseille)

7264:

Sade, A. Isomorphisme d'hypergroupoïdes isotopes. Pacific J. Math. **9** (1959), 583–584.

Let H be a hypergroupoid with a two-sided unit element. If H is isotopic to an associative hypergroupoid G, then H is isomorphic to G. The isomorphism can be explicitly expressed in terms of the mappings of the isotopy.

H. B. Mann (Columbus, Ohio)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 7270, 7477, 7500.

7265:

Solodovnikov, A. S. The structure of the centers of universal coverings for non-compact Lie groups. Dokl. Akad. Nauk SSSR **129** (1959), 272–275. (Russian)

The author determines for each classical real non-compact group its centre (which, he observes, suffices in principle to determine all locally isomorphic groups). He does this by finding in each case a maximal compact subgroup and applying a theorem of E. Cartan [cf. G. D. Mostow, Bull. Amer. Math. Soc. **55** (1949), 969–980; MR **11**, 326].

P. M. Cohn (Manchester)

7266:

Oniščik, A. L. Insertions between transitive compact groups of transformations. Dokl. Akad. Nauk SSSR **129** (1959), 261–264. (Russian)

Let $\mathfrak{G}$ be a connected compact Lie group, acting transitively and effectively on a manifold X. The author completes the classification of all subgroups of $\mathfrak{G}$ which act transitively on X [cf. his paper, same Dokl. **124** (1959), 520–523; MR **21** #2707]. The problem is equivalent to finding all decompositions $(\mathfrak{G}, \mathfrak{G}', \mathfrak{G}'')$, where $\mathfrak{G}'$, $\mathfrak{G}''$ are subgroups of $\mathfrak{G}$ such that $\mathfrak{G}=\mathfrak{G}'\mathfrak{G}''$, or equivalently, finding all decompositions $\mathfrak{g}=\mathfrak{g}'+\mathfrak{g}''$ of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. A list of all possible decompositions (up to conjugacy) is given. The author also gives a topological criterion for a triple $(\mathfrak{G}, \mathfrak{G}', \mathfrak{G}'')$ to be a decomposition, where $\mathfrak{G}$ is a connected compact Lie group and $\mathfrak{G}'$, $\mathfrak{G}''$ are connected closed subgroups. This is expressed as the exactness of a certain sequence of spaces of primitive elements of the cohomology rings (with real coefficients) of $\mathfrak{G}$, $\mathfrak{G}'$, $\mathfrak{G}''$ and $\mathfrak{U}_0$, the connected identity-component of $\mathfrak{G}' \cap \mathfrak{G}''$; it then follows that $(\mathfrak{G}, \mathfrak{G}', \mathfrak{G}'')$ is a decomposition if and only if $P(\mathfrak{G}, t) \cdot P(\mathfrak{U}_0, t) = P(\mathfrak{G}', t) \cdot P(\mathfrak{G}'', t)$, where $P(X, t)$ is the Poincaré polynomial of the manifold X.

P. M. Cohn (Manchester)

7267:

Kanno, Tsuneo. On replicas. Tôhoku Math. J. (2) **11** (1959), 287–298.

If D, D' are elements of the Lie algebra of an algebraic group G, define D' to be a replica of D if D' annuls each rational function on G that is annulled by D. It is shown that this agrees with the definition previously given for the case of characteristic zero, where the condition is that D' be contained in the Lie algebra of each algebraic subgroup of G whose Lie algebra contains D. It is also

shown, for linear groups over fields of arbitrary characteristic, that the new definition agrees with Chevalley's original definition involving tensor invariants.

M. Rosenlicht (Berkeley, Calif.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

7268:

Curtis, M. L. Self-linked subgroups of semigroups. Amer. J. Math. **81** (1959), 889–892.

If 3-space is provided with a continuous associative multiplication and if C is a circle subgroup, then C is not self-linked; if, moreover, C is tame, then it cannot be knotted. *A. D. Wallace* (New Orleans, La.)

7269:

Koch, R. J. Arcs in partially ordered spaces. Pacific J. Math. **9** (1959), 723–728.

This paper proves the existence of an arc in partially ordered spaces, where arc means a continuum irreducibly connected between two points. Among the applications are (1) a compact connected topological semigroup with unit contains an arc in every neighborhood of the unit, and (2) a compact connected topological semigroup with zero, each of whose elements is idempotent, is arcwise connected. All spaces are Hausdorff, but not necessarily metric. An example is given to show in (1) that there may be no arc including the unit.

D. Montgomery (Princeton, N.J.)

7270:

Maurer, I. Sur la topologisation des anneaux. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 177–180. (Romanian. Russian and French summaries)

The author observes that the method which he used [Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 265–272; Com. Acad. R. P. Romîne **8** (1958), 5–11; Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (**50**) (1958), 55–59; MR **18**, 907; **21** #4988, #4989] to introduce a topology into the group of permutations of an arbitrary infinite set—by first defining suitably the limit of a sequence of such permutations—provides in fact a method for topologizing an arbitrary abstract group, since this can be embedded in the group of permutations of its elements. He now shows how a precisely analogous procedure can be used to topologize first the ring of endomorphisms of an abelian group, and thence any abstract ring. It is mentioned in a footnote that there exists another method of topologizing abstract rings, due to T. Szele and as yet unpublished.

I. M. H. Etherington (Edinburgh)

FUNCTIONS OF REAL VARIABLES

7271:

Watanabe, Yoshikatsu; and Ichijô, Yoshihiro. Zur Laplaceschen asymptotischen Formel. J. Gakugei Tokushima Univ. Math. **9** (1958), 1–17; addendum, **10** (1959), 39.

The authors establish by direct evaluation the asymptotic behaviour of a class of integrals of the form $\int_a^b \varphi(x)\,[f(x)]^n dx$ for large n. The main assumption, besides differentiability requirements, is that $f(x)$ assumes its maximum at a single point.

A. Dvoretzky (Jerusalem)

7272:

Kahane, Jean-Pierre. Sur l'exemple, donné par M. de Rham, d'une fonction continue sans dérivée. Enseignement Math. (2) **5** (1959), 53–57.

The author remarks on some simple properties of a non-differentiable function given by G. de Rham [Enseignement Math. (2) **3** (1957), 71–72; MR **19**, 20]. He considers the nature of the set of points in which the function has a maximum or a minimum and also considers the modulus of continuity of similar functions.

U. S. Haslam-Jones (Oxford)

7273:

★Martinot-Lagarde, André. Sur la distinction entre l'existence de la dérivée et une propriété analogue. Actes du colloque de calcul numérique, Périgueux, 1957, pp. 15–17. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii+87 pp. 1800 francs.

A property D of a function $f(x)$ of a real variable is considered of which it is known that the existence of a derivative $f'(x)$ implies that $f(x)$ has property D, whilst in simple cases D implies that $f'(x)$ exists. The author produces an example of a function with property D for which $f'(x)$ does not exist. *U. S. Haslam-Jones* (Oxford)

7274:

Hoang, Tuĭ [Hoãng, Tuy]. Symmetry of the contingent of the graph of a measurable function. Dokl. Akad. Nauk SSSR **126** (1959), 946–947. (Russian)

The fractional density analogues of certain derivatives, previously studied by the author [same Dokl. **126** (1959), 37–40; MR **21** #5000], are discussed, not only for one-sided density, but also for symmetric density. The author announces for them two propositions which extend the symmetry relations of Denjoy and Mrs. Young.

L. C. Young (Madison, Wis.)

7275:

Kazakevič, V. V. Monotonically stable invariant points. Dokl. Akad. Nauk SSSR **126** (1959), 287–290 (Russian); translated as Soviet Physics. Dokl. **4**, 574–577.

Let $P_p: (x_1^{(p)}, x_2^{(p)}, \cdots, x_n^{(p)})$, a point in Euclidean n-space, be carried into the point P_{p+1} by the transformation T; $p=1, 2, \cdots$. Let P^* be a fixed point of this transformation and $\rho(P_p)$ be the distance from P_p to P^*. The transformation is said to be monotonically stable near P^* if there exists an integer $N>0$ such that $\rho(P_{p+1})<\rho(P_p)$ for $p=N, N+1, \cdots$ and if $\lim \rho(P_p)=0$ as $p\to\infty$. A sufficient condition for the transformation to be monotonically stable near P^* is that a certain matrix be positive definite. The corresponding positive definite quadratic form is obtained by expanding in a Taylor series the difference $[\rho(P_p)]^2-[\rho(P_{p+1})]^2$. *M. Marden* (Madison, Wis.)

7276:

Marcus, S. Sur un théorème de G. Szekeres, concernant les fonctions monotones et convexes. Canad. J. Math. **11** (1959), 521–526.

Let f be called of type K if it can be written in the form $f(x)=\psi(\varphi(x))$ with φ convex and strictly increasing and ψ concave and strictly increasing. According to Szekeres [Proc. Amer. Math. Soc. **7** (1956), 351–353; MR **18**, 23], any strictly increasing f with a continuous second derivative is of type K. The author first shows that, for a function of type K, $f^+(x)$ and $f^-(x)$ (the right-hand and left-hand derivatives) exist, are strictly positive, and have derivatives almost everywhere, integrable on compact subsets of the domain of f. Hence $f(x)=x^3$ contradicts Szekeres's theorem (whose proof tacitly used $f'(x)\neq 0$). The author then shows that these necessary conditions become sufficient if strengthened to require that f^+ has a derivative everywhere instead of almost everywhere (in which case f' and f'' exist everywhere).

R. P. Boas, Jr. (Evanston, Ill.)

7277:

Reid, William T. Variational aspects of generalized convex functions. Pacific J. Math. **9** (1959), 571–581.

Let $r(x)$, $p(x)$, $q(x)$ be defined in $a<x<b$, $r(x)>0$, and consider the self-adjoint differential equation

$$(1) \qquad L(y) \equiv (ry'+qy)'-(qy'+py) = 0$$

having the property that whenever $a<x_1<x_2<b$ there exists exactly one solution $y(x)=y(x;x_1,y_1,x_2,y_2)$ of (1) such that $y(x_i)=y_i$ $(i=1,2)$, where y_1 and y_2 are arbitrary real numbers. A function $u(x)$ is said to be sub-L on (a,b) if $u(x)\leqq y(x;x_1,u(x_1),x_2,u(x_2))$ for $x_1\leqq x\leqq x_2$. It has been proved by the reviewer [Bull. Amer. Math. Soc. **55** (1949), 563–572; MR **10**, 686] and by Bonsall [Quart. J. Math. Oxford Ser. (2) **1** (1950), 100–111; MR **12**, 83] that in order that u of class C^2 be sub-L it is necessary and sufficient that $L(u)\geqq 0$, a fact that generalizes a well known property of convex functions. The main theorem of the present paper is that sub-L functions can also be characterized by the property of affording a minimum for a certain unilateral variation problem. Let $a<c<d<b$ and denote by $\Gamma(c,d)$ the set of functions $\eta(x)$ that are absolutely continuous on the compact interval $[c,d]$ and such that $\eta'(x)\in L^2(c,d)$. By $\Gamma_0(c,d)$ we mean those functions of $\Gamma(c,d)$ vanishing at c and d, and by $\Gamma_0^-(c,d)$ those functions $\eta(x)$ of $\Gamma_0(c,d)$ such that $\eta(x)\leqq 0$ on $[c,d]$. Call

$$I(\eta;c,d) = \int_c^d (r\eta'^2+2q\eta\eta'+p\eta^2)dx.$$

Theorem: $u(x)$ is sub-L on (a,b) if and only if for each compact sub-interval $[c,d]$ of (a,b) the function $u(x)\in\Gamma(c,d)$ and $I(\eta;c,d)\geqq I(u;c,d)$ for $\eta-u\in\Gamma_0^-(c,d)$.

M. M. Peixoto (Rio de Janeiro)

7278:

Boas, R. P., Jr. Representations for completely convex functions. Amer. J. Math. **81** (1959), 709–714.

A completely convex function $f(x)$ on $[0,1]$ is an infinitely differentiable function such that $(-1)^n f^{(2n)}(x)\geqq 0$. The author proves that $f(x)$ is completely convex on $[0,1]$ if and only if it has the form

$$f(x) = \sin \pi x\left\{c+\int_0^\infty \phi(t)[\cosh \pi t+\cos \pi x]^{-1}dt + \int_0^\infty \psi(t)[\cosh \pi t-\cos \pi x]^{-1}dt\right\}$$

for some non-negative constant c and some ϕ and ψ that are even entire functions of exponential type π with non-negative Maclaurin coefficients. He also obtains necessary and sufficient conditions that A. Wintner's expression [Amer. J. Math. **80** (1958), 125–130; MR **20** #92] shall represent a completely convex function. The results depend on Widder's characterization of minimal completely convex functions by their Lidstone series expansions.

L. A. Rubel (Urbana, Ill.)

MEASURE AND INTEGRATION

See also 7426, 7459a–b.

7279:

★Rosenthal, Arthur. Introduction to the theory of measure and integration. Notes by R. P. Smith. The Department of Mathematics, The Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1955. 125+ iv pp.

This volume embodies the lectures given by A. Rosenthal to an NSF Summer Institute for Teachers of Mathematics. It starts on the elementary level of Riemann integrals of functions defined on a linear interval, and ultimately presents material on Denjoy and Perron integration, as well as a theory of abstract additive functions on additive families of sets and a sophisticated approach to the Lebesgue type of integral of a function on an abstract space, due to H. Hahn, all in the space of 125 mimeographed pages. The table of contents is: 1. The Riemann integral; 2. The Jordan content; 3. The Lebesgue measure; 4. Application of content and measure to the Riemann integral; 5. The Lebesgue integral; 6. Measurable functions; 7. Properties of Lebesgue integrals; 8. Relation between integration and differentiation; 9. Stieltjes integrals; 10. Set functions; 11. Integration in general spaces. There is added a set of general references, as well as particular references to individual chapters which enable the reader to look up proofs omitted in the text, as well as build on the basic ideas presented in the lectures.

T. H. Hildebrandt (Ann Arbor, Mich.)

7280:

Letta, Giorgio. Il problema di Vitali-Lusin negli spazi perfettamente normali. Ricerche Mat. **8** (1959), 128–137.

Cette note apporte quelques compléments aux travaux de F. Cafiero [Matematiche. Catania **11** (1956), 144–162 (1957); MR **20** #2412] et F. Bertolini [Ricerche Mat. **6** (1957), 288–306; MR **20** #3250] sur l'équivalence entre mesurabilité et quasi-continuité. S: ensemble abstrait. f: fonction numérique et finie sur S. $\mathcal{F}$: famille de sous-ensembles de S contenant S et l'ensemble vide $\emptyset$, close (stable selon N. Bourbaki) pour l'union finie et l'intersection dénombrable. $\mathcal{G}$: famille des ensembles $G=S-F$ où $F\in\mathcal{F}$. L'espace "topologique" ("pseudo-topologique" suivant O. M. Nikodym) $(S,\mathcal{F})$ est dit "parfaitement normal" si deux ensembles de $\mathcal{F}$ disjoints quelconques

peuvent être séparés par deux ensembles de $\mathscr{G}$ et si tout ensemble $F \in \mathscr{F}$ est l'intersection d'une sous-famille dénombrable de $\mathscr{G}$. Deux propriétés pour $(S, \mathscr{F})$ sont formulées et leur équivalence avec la normalité parfaite est démontrée. μ: mesure à valeurs finies ≥ 0, définie sur une σ-algèbre booléenne $\mathscr{L}$ de sous-ensembles de S, d'unité S. $\mathscr{M}$: ensemble des fonctions f μ-mesurables. $(S, \mathscr{F})$ est supposé désormais parfaitement normal. Pour $\mathscr{F} \subseteq \mathscr{M}$, μ° désigne la complétion de la restriction de μ à la famille des ensembles de Borel de $(S, \mathscr{F})$. La fonction f est dite "quasi-continue" ["quasi-continue selon Vitali-Lusin"] si à tout $\varepsilon > 0$ correspond un ensemble $L \in \mathscr{L}$ [un ensemble $L \in \mathscr{L} \cap \mathscr{F}$] tel que $\mu(S - L) < \varepsilon$ et que la restriction de f à L soit continue relativement à la topologie induite par $\mathscr{F}$ dans L. $\mathscr{Q}$ [$\mathscr{Q}^\times$]: ensemble des fonctions quasi-continues [quasi-continues selon Vitali-Lusin]. Théorème: $(\mathscr{M} = \mathscr{Q}) \Leftrightarrow ((\mu$ est complète) & $(\mathscr{F} \subseteq \mathscr{L})$ & (tout $M \in \mathscr{M}$ est μ-équivalent à un ensemble de Borel)). $(\mathscr{M} = \mathscr{Q}^\times) \Leftrightarrow ((\mathscr{F} \subseteq \mathscr{L})$ & ($\mathscr{L}$ = famille des ensembles μ°-mesurables)). {Remarques du rapporteur: (1) L'hypothèse de normalité parfaite ne permet pas l'application du théorème précédent aux mesures de Radon générales. Des résultats obtenus sans cette hypothèse se trouvent chez H. M. Schaerf [Portugaliae Math. **6** (1947), 33–44; errata, 66; MR **9**, 18] et chez O. Haupt, G. Aumann, et C. Pauc [*Integralrechnung*, Walter de Gruyter, Berlin, 1955; MR **17**, 1066; errata **18**, 1118; **19**, 1431; §§ 8.1 et 8.2]. (2) La mesure μ n'intervient en fait que par la famille des ensembles de mesure nulle.} *Chr. Pauc* (Nantes)

7281:

Mahowald, Mark. On a paper of Maurice Sion. Canad. J. Math. **11** (1959), 409–415.

M. Sion [Trans. Amer. Math. Soc. **83** (1956), 205–221; MR **19**, 21] defines a class M_2 of measures and proves that Lebesgue measure is not in M_2. The author gives a new proof of this result and discusses the properties of M_2 with respect to open regular measures.

U. S. Haslam-Jones (Oxford)

7282:

Marcus, S. La mesure de Jordan et l'intégrale de Riemann dans un espace mesuré topologique. Acta Sci. Math. Szeged. **20** (1959), 156–163.

The present paper generalizes and suggests improvements to preceding work, principally of M. Nicolescu, O. Frink, J. Lipiński, and S. Marcus. In his development, the author uses vertical partitions. The definition of Jordan measurability (due to O. Haupt and C. Pauc) is the following. S is a topological space, k is a σ-algebra of subsets of S which includes all Borel sets in S, μ is a complete finite measure on k. Then $X \subset S$ is said to be Jordan measurable if $\mu(\mathrm{Fr}\, X) = 0$ where $\mathrm{Fr}\, X$ stands for the frontier of X. The development contains six theorems stating, for the most part, classic properties of the Riemann integral (for example, a bounded real function f is R-integrable if and only if the set Δ of points of discontinuity of f has μ measure zero; or if and only if f is J-measurable). The following theorem of Lusin type is new. A necessary and sufficient condition that for every $\varepsilon > 0$ there be a J-measurable set E_ε such that $\mu(E_\varepsilon) > \mu(S) - \varepsilon$ and that the restriction of f to E_ε be continuous is that Δ be J-measurable and that $\mu(\Delta) = 0$.

E. R. Lorch (New York, N.Y.).

7283:

Kubota, Yôto. On the definition of Cesàro-Perron integrals. Tôhoku Math. J. (2) **11** (1959), 266–270.

The note answers a question posed by R. L. Jeffery in his review of G. Sunouchi and M. Utagawa, same J. (2) **1** (1949), 95–99, in MR **1**, 90, by showing that their Cesàro-Perron integral is equivalent to the Cesàro-Denjoy integral of W. L. C. Sargent [Proc. London Math. Soc. (2) **17** (1940), 202–247; MR **3**, 228], which in turn is equivalent to the CP_0 integral of J. C. Burkill.

T. H. Hildebrandt (Chapel Hill, N.C.)

7284:

Hayes, Charles A., Jr. Some principles underlying the construction of measures. Canad. J. Math. **11** (1959), 601–613.

S, T: fundamental sets. A, B: generic subsets of S. $\mathfrak{A}$: family of all subsets of S. Measure on S (Caratheodory measure function): non-negative possibly infinite function Φ defined on $\mathfrak{A}$ such that, for any countable subfamily $\mathfrak{F}$ of $\mathfrak{A}$ covering A, $\Phi(A)$ is $\leq \sum \Phi(X)$, the summation extending over $\mathfrak{F}$. $\mathfrak{F}_0$: subfamily of $\mathfrak{A}$. $\mathbf{M}(A, t)$: collection of countable subfamilies of $\mathfrak{F}_0$, defined for all $A \in \mathfrak{A}$ and $t \in T$. f: non-negative finite-valued function defined on $\mathfrak{F}_0$. $f_t(A) = \inf \sum f(G)$, $G \in \mathfrak{G}$, $\mathfrak{G} \in \mathbf{M}(A, t)$. $\bar{f}(A) = \sup f_t(A)$, $t \in T$. Conditions on $\mathbf{M}(A, t)$ are stated under which $\bar{f}$ is a measure; they are assumed in §§ 3, 4, 5. § 3: Extra restrictions on $\mathfrak{F}_0$ and f are formulated securing the Caratheodory $\bar{f}$-measurability of the $\mathfrak{F}_0$-sets. In § 4 an abstract separation relation R for subsets of S and an axiom are introduced implying the additivity of $\bar{f}$ on R-separated sets. § 5 gives a condition for the equality of $\bar{f}$ and f on $\mathfrak{F}_0$. § 6 contains applications: (1) S is a metric space, T is the set P of positive real numbers, $\mathbf{M}(A, t)$ is the collection of countable coverings of A by $\mathfrak{F}_0$-sets of diameter $< t$, $R = \{(A, B)$: $\mathrm{dist}(A, B) > 0\}$. Special case: The $\mathfrak{F}_0$-sets are open, $f(A)$ is a fixed positive power of $\mathrm{diam}\, A$. (2) Halmos' extension of a content to a Radon measure. (3) S is provided with a measure Φ, $T = P$, $\mathbf{M}(A, t)$ is the collection of countable subfamilies of $\mathfrak{F}_0$ covering A but for a set of measure $< t$. {Remark by the reviewer: Another application is supplied by the L. Cesari [*Surface area*, Princeton Univ. Press, 1956; MR **17**, 596] measure function Φ derived from the Lebesgue area where $R = \{(A, B)$: A and B are separated by whole open sets (i.e. $\mathfrak{G}_0$-sets)$\}$.} *Chr. Pauc* (Nantes)

7285:

Nakanishi, Shizu. Sur le théorème de Fubini et les suites fondamentales. Proc. Japan Acad. **35** (1959), 161–166.

Théorème de Fubini pour les intégrales au sens de Denjoy en utilisant la méthode des espaces rangés.

A. Appert (Angers)

7286:

Kelley, J. L. Measures on Boolean algebras. Pacific J. Math. **9** (1959), 1165–1177.

For any non-empty subset $\mathscr{B}$ of a Boolean algebra $\mathscr{A}$, the intersection number is by definition $I(\mathscr{B}) = \inf\{i(S)/n(S)$: S a finite sequence of elements of $\mathscr{B}\}$, where $n(S)$ denotes the number of terms in S, and $i(S)$ is the maximum number of terms of S having a non-zero intersection. Let M denote the class of finitely additive

normalized measures on $\mathscr{A}$. Principal results: (1) $I(\mathscr{B}) \geq \inf\{m(B): B \in \mathscr{B}\}$ for each $m \in M$; equality holds for some $m \in M$. (2) A strictly positive measure $m \in M$ exists if and only if $\mathscr{A} - \{0\}$ is a countable union of sets $\mathscr{B}_n$ with $I(\mathscr{B}_n) > 0$. (3) A complete Boolean algebra $\mathscr{A}$ admits a strictly positive countably additive measure if and only if $\mathscr{A}$ is weakly countably distributive (w.c.d.) and admits a strictly positive measure. (4) A complete Boolean algebra that satisfies the countable chain condition is w.c.d. if and only if each set of first category in its Stone space is nowhere dense. (5) If the Souslin hypothesis is false there exists a linear continuum whose algebra $\mathscr{A}$ of regular open sets is complete, w.c.d., satisfies the countable chain condition, admits no strictly positive measure, and each $m \in M$ vanishes for the elements of some sequence whose sup is $\mathscr{A}$. (6) If p is a non-negative, monotone real function such that $p(A)+p(B) \geq p(A \cup B)+p(A \cap B)$ on $\mathscr{A}$, and if m is a measure on a subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $m(B) \leq p(B)$ on $\mathscr{B}$, then m can be extended to a measure n on $\mathscr{A}$ such that $n(A) \leq p(A)$ on $\mathscr{A}$.

J. C. Oxtoby (Bryn Mawr, Pa.)

7287:

Glaeser, Georges. Intégration d'une forme différentielle le long de certaines courbes non rectifiables. Trans. Amer. Math. Soc. **93** (1959), 169–183.

Let Γ be a simple Jordan arc in the plane and $\omega = P(x, y)dx + Q(x, y)dy$ where P and Q are functions of class m (i.e., m-times continuously differentiable). Consider the problem whether there is a function $f(x, y)$ of class $m+1$ such that $df = \omega$ up to order m on Γ in the sense that every partial derivative of f of order $i = 1, \cdots, m+1$ agrees on Γ with the corresponding partial derivative of order $i-1$ of P or Q. An example shows that for $m = 1$ the necessary condition $\partial P/\partial y = \partial Q/\partial x$ on Γ is not sufficient. An affirmative result for any $m \geq 1$ is obtained for a special class of arcs Γ. Each such Γ is locally similar to itself. This supplements a theorem of M. Sion [Trans. Amer. Math. Soc. **77** (1954), 179–201; MR **16**, 344]. Results are also found for the case when P and Q are of class $r > m$.

W. H. Fleming (Providence, R.I.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 7368, 7376, 7534.

7288:

★Векуа, И. Н. Обобщенные аналитические функции. [Vekua, I. N. **Generalized analytic functions.**] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 628 pp. 18.25 rubles.

By generalized analytic functions are meant functions of two real variables which obey elliptic differential equations analogous to the Cauchy-Riemann equations. Throughout this review, the functions and equations referred to are in two independent real variables: elliptic systems are systems of two first order equations.

The chapter headings are: (I) Certain classes of functions and operators. (II) Reduction of positive differential quadratic forms to canonical form. Beltrami equation. Geometrical application. (III) Bases of the general theory of generalized analytic functions. (IV) Boundary problems. (V) General theory of infinitesimal deformations of surfaces. (VI) Problems of the moment-free theory of shells.

Chapter I is of a preliminary nature; a number of spaces of functions are defined with arguments in an open domain G, with boundary Γ, of the complex plane E; properties of such functions are stated, with proof or references to the literature. In particular the operator $T = T_G$ is defined, where $(Tf)(z) = -\pi^{-1} \int_G f(\zeta)(\zeta - z)d\xi d\eta$ $(\zeta = \xi + i\eta)$ so that if $\partial_{\bar{z}} w = f$, where $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, then $w(z) = \Phi(z) + (Tf)(z)$ where $\Phi(z) = (2\pi i)^{-1} \int_\Gamma w(\zeta) \times (\zeta - z)^{-1} d\zeta$. The generalized Sobolev derivatives are discussed for this case: $g = \partial_{\bar{z}} f$ if f and g are summable over G, and $\int_G (g\partial_{\bar{z}}\varphi + f\varphi)dxdy = 0$ for every φ differentiable in G and vanishing outside some closed set inside G.

Chapter II deals with the reduction of the form $adx^2 + 2bdxdy + cdy^2$, where a, b, c are functions of x, y, and $ac - b^2 > 0$, to the form $\Lambda(du^2 + dv^2)$. The problem is reduced to the solution of a Beltrami equation and then to the form of an integral equation, and under appropriate conditions solutions are shown to exist locally, over all G, and over all E. The applications to the problem of introducing isometric (i.e., isothermal) coordinates on a surface are discussed. The method is extended to discuss the reduction of a general elliptic system to the canonical form $u_x - v_y + au + by = f$, $u_x + v_y + cu + dv = g$, and of a second order elliptic equation to the form with second order terms $u_{xx} + u_{yy}$.

Chapter III is the kernel of the book. It discusses solutions of the equation $Cw = F$ where $Cw = \partial_{\bar{z}} w + Aw + B\bar{w}$, in a domain G. w is a generalized solution of this in the neighbourhood of z_0 if it has a generalized derivative $\partial_{\bar{z}} w$ summable over a neighbourhood of z_0 and $Cw = F$ almost everywhere in that neighbourhood. w is called a generalized solution of $Cw = F$ in G if it is a generalized solution of $Cw = F$ in the neighbourhood of all save a discrete set of points of G; if there are no exceptional points w is called a regular solution. The generalized solutions of $Cw = 0$ are called generalized analytic functions. Existence of such solutions is discussed, and their properties examined, under the hypotheses that A, B, F belong to specific classes $L_p(G)$. Integral equations for the solutions are given: if $Cw = F$, then $w - T(Aw + B\bar{w}) = \Phi(z) + TF$, where Φ is an analytic function in G, and if this integral equation has a solution with $Aw + B\bar{w} \in L(G)$, then it is a solution of $Cw = F$. The following fundamental theorem is proved: if w is a generalized analytic function then $w(z) \exp \omega(z)$ is analytic in G, where

$$\omega(z) = -T[A(z) + B(z)|w(z)|/w(z)].$$

Analogues of the maximum modulus principle, the principle of uniform convergence for analytic functions, and of Liouville's theorem are proved. The pseudoanalytic functions of Bers are discussed. The author proves the result (found independently by himself and by Bers [*Theory of pseudoanalytic functions*, Institute for Mathematics and Mechanics, New York Univ., New York, 1953; MR **15**, 211]) that for any function $\Phi(z)$ analytic in G and $t \in G$ there is a unique generalized analytic function $w = \mathfrak{R}_t(\Phi)$ such that $w(z)/\Phi(z)$ is continuous in G and can be extended continuously to E, is not zero in E, and is equal to 1 at t. An analogue of Cauchy's integral formula is proved in which the kernel $(t-z)^{-1}$ is replaced by kernels constructed from the functions $\mathfrak{R}_t[(t-z)^{-1}]$, $\mathfrak{R}_t[i(t-z)^{-1}]$; this is used to discuss continuation of generalized analytic functions. Generalized power series,

in terms of functions $\Re_l[(z-z_0)^n]$ are discussed; analogues of Taylor and Laurent expansions are shown to exist. A further section extends these results to general elliptic systems.

Chapter IV discusses boundary problems, particularly those of the type termed generalized Riemann-Hilbert problems by Mushelišvili, for elliptic equations and systems. A number of special cases are discussed, and problems involving both the function and its derivatives in the boundary conditions are also treated. The chapter ends with a note by Boyarskiĭ on special cases of the problem.

Chapter V deals with small deformations of piecewise regular inextensible surfaces of positive curvature. The differential equations of the problem are deduced ab initio, and treated for both closed surfaces and surfaces with boundaries, and also for surfaces subject to constraints. Conditions for rigidity of surfaces are found and applied to discuss rigidity of surfaces of revolution. For non-rigid surfaces, conditions in order that the possible deformations form a finite dimensional manifold are given; and for surfaces which are rigid when subject to constraints, conditions for the constraints to be overdetermined (incorrect, in the author's terminology) and the degree of overdetermination are discussed.

Chapter VI deals with the theory of thin elastic shells, particularly with the theory of equilibrium with zero bending moment (i.e., the membrane theory of shells, in English terminology). The theory is developed ab initio, and the results of previous chapters are used to discuss the existence and number of solutions. For a convex shell with m holes there are $3m-3$ linearly independent solutions to the equilibrium equations, in the membrane case, given assigned loadings along the normals or other lines inclined to the tangents at the holes. The realizability of these solutions in practice is discussed, both for surfaces with positive curvature and others: conditions for the momentfree solution to be stable are found, and as these are usually not satisfied the conditions for approximately momentfree conditions of equilibrium to exist are discussed.

The book ends with a bibliography containing extensive references to Soviet and other work in the fields discussed. It also has historical surveys of the literature in the text. It is well and clearly written, though unfortunately it has no index either of subjects or of notations; the latter would be particularly helpful to the reader. *J. L. B. Cooper* (Cardiff)

7289:

Scilard, K. S. Extension of the Fatou theorem to a class of continuous mappings. Dokl. Akad. Nauk SSSR **127** (1959), 278–280. (Russian)

The following theorem of the Fatou type is established. Let the following three conditions be satisfied by the function $f=u+iv$ in the domain

$$E = \{re^{i\varphi} | \varphi_1 < \varphi < \varphi_2,\ r_1 \leqq r < 1\}:$$

(1) $\zeta=f(z)$ defines locally an interior transformation of the set E. Moreover, the mapping is locally univalent except for a countable set of points not having a cluster point in the interior of E. (2) The integral

$$I_E = \iint_E (|D(x,y)|)^{1/2} dx dy$$

is finite. Here $D(x,y)$ denotes the Jacobian of the transformation $u=u(x,y)$, $v=v(x,y)$. (3) There exists a constant C, $1 \geqq C > 0$, such that, for a point z_0 in a neighbourhood of which the map f is univalent, one has

$$\liminf_{\rho \to 0} \frac{\min_{|z-z_0|=\rho} |f(z_0)-f(z)|}{\max_{|z-z_0|=\rho} |f(z_0)-f(z)|} \geq C.$$

Then $\lim_{r\to 1} f(re^{i\varphi})$ exists for almost all φ in $\{\varphi_1 < \varphi < \varphi_2\}$.

The second condition of the hypothesis finds its source in Carathéodory, Nachr. K. Ges. Wiss. Gött. Math. Phys. Kl. **1913**, 509–518, where a hypothesis of this type is used in a special case. *M. H. Heins* (Urbana, Ill.)

7290:

Ozawa, Mitsuru. On an approximation theorem in a family of quasiconformal mappings. Kōdai Math. Sem. Rep. **11** (1959), 65–76.

The paper is devoted to the problem of extremal quasiconformal mapping with given boundary correspondence. With the help of a useful approximation lemma and the customary length-area argument, it is first shown that an affine mapping of a region with finite area is extremal for the boundary correspondence which it induces. It is then concluded that the Teichmüller mapping is extremal in the whole class of mappings of the universal covering surface which agree on the boundary. The method does not prove uniqueness.

Some additional results are complicated to state but would probably be valuable in a systematic attack on the whole problem. *L. Ahlfors* (Cambridge, Mass.)

7291:

Gheorghiu, Octav Em. Sur les fonctions hypercomplexes de plusieurs variables monogènes au sens de V. S. Feodoroff et M. Nicolescu. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 393–400.

The author defines "monogeneity F-N" for a function of two (or more) hypercomplex variables $\omega_l = x_l + y_l\theta + z_l\theta^2$ where $\theta^3=1$, and shows some elementary properties. This is an extension of the ideas of V. S. Feodoroff [Gr. C. Moisil, Acad. R. P. Romîne. Bul. Ştiinţ. A. **1** (1949), 959–964; MR **13**, 455] and of Miron Nicolescu [Lucrările sesiunii generale ştiinţifice ale Academiei R. P. Romîne din 2–12 iunie 1950, pp. 187–194, Edit. Acad. R. P. Romîne]. *J. A. Ward* (Holloman Air Force Base, N.M.)

7292:

Rahman, Q. I. Inequalities for polynomials. Proc. Amer. Math. Soc. **10** (1959), 800–806.

The author establishes a number of connections among known inequalities for polynomials and their derivatives, with and without restrictions on the location of the zeros. New results include the following. Let $p(z)$ be a polynomial of degree n with $\max|p(z)|=1$ on $|z|=1$. (1) If all the zeros of $p(z)$ are in $|z| \geq 1$ and at least one is in $|z|>1$, then there exists $\delta>0$ such that $\max|p(z)| > (1-\rho^n)/2$ on $|z|=\rho$ if $1-\delta<\rho<1$. (2) If $\max|p(z)| \leq (1-R_m^n)/2$ for $|z|=R_m>1$ with $R_m \downarrow 1$, then $p(z)$ does not have all its zeros in $|z|<1$. Analogous results hold for the mean value

of $|p(z)|$. In particular, if $p(z)$ has no zeros in $|z|<1$ then, for some $\delta>0$, $1-\delta<\rho<1$,

$$\int_0^{2\pi}|p(\rho e^{i\varphi})|d\varphi>\{1-\tfrac{1}{4}\pi(1-\rho^n)\}\int_0^{2\pi}|p(e^{i\varphi})|d\varphi.$$

R. P. Boas, Jr. (Evanston, Ill.)

7293:

Parodi, Maurice. Sur la localisation des zéros des polynomes dont les coefficients ont des valeurs voisines. C. R. Acad. Sci. Paris **249** (1959), 1307–1308.

It is well known th ıt the zeros of a given polynomial

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$$

are the characteristic values of the matrix

$$F=\begin{pmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\cdot&\cdot&\cdot&\cdots&\cdot&\cdot\\0&0&0&\cdots&0&1\\-a_n&-a_{n-1}&-a_{n-2}&\cdots&-a_2&-a_1\end{pmatrix}.$$

Let $F'=BFB^{-1}$ where $B=(b_{ij})$ with $b_{ij}=1$ for $j\leqq i$ and $b_{ij}=0$ for $j>i$. Then applying Hadamard's Theorem to the determinant of the matrix $F'-zI$, the author proves that all the zeros of $f(z)$ lie in the union of the two disks

$$|z|\leqq 2 \text{ and } |1-a_1-z|\leqq |a_{n-1}-a_n-1|+|a_{n-2}-a_{n-1}|+\cdots+|a_1-a_2|.$$

M. Marden (Madison, Wis.)

7294:

Ponting, F. W. The location of singularities on the circle of convergence of gap series. II. Collect. Math. **10** (1958), 137–184.

[Pour la partie I, voir Quart. J. Math. Oxford Ser. (2) **4** (1953), 19–35; MR **14**, 1074.] Supposons que $\varphi(z)=\sum c_nz^n$ admette sur son cercle de convergence un nombre pair $k=2l$ de points singuliers isolés, non critiques de type exponentiel fini. Si la densité supérieure des coefficients nuls de la série est supérieure à $1-1/l$, ces points singuliers sont situés sur k sommets d'un polygone régulier. Le nombre de côtés de ce polygone est k, $k+w$, où w est un facteur de k; $k+2w'$, où $3w'=k$; ou alors jl, où j est un entier supérieur à 3. *S. Mandelbrojt* (Paris)

7295:

Wyman, Max. The asymptotic behavior of the Laurent coefficients. J. Soc. Indust. Appl. Math. **7** (1959), 325–342.

Let $G(z)=\sum_{-\infty}^{\infty}a_nz^n$ be regular in the annulus $a<|z|<b$. The author describes a method by which an asymptotic expansion for a_n as $n\to\infty$ may be obtained from the Cauchy formula

$$(1)\qquad a_n=(2\pi i)^{-1}\int_C G(z)z^{-(n+1)}dz,$$

where C is a suitable contour enclosing $z=0$. The method requires that there be a finite number of paths $\theta=\arg z=\theta_k(|z|)$ on which $|G(re^{i\theta})|$ has a local maximum, and in certain neighbourhoods of which an asymptotic expansion of the form

$$\log G(re^{i\theta})\approx\sum_{n=0}^{\infty}a_{n,k}(r)(\theta-\theta_k)^n$$

is valid and such that the parts of C outside these neighbourhoods contribute only relatively little to the integral in (1). The method has been developed by a number of previous writers including Darboux, Hardy and Littlewood, Szekeres, Moser, the author and the reviewer. Examples given include, for instance,

$$G(z)=(1-z)^{-1}\exp[1/2(1-z^2)],$$

where

$$a_n=\frac{e^{n^{1/2}+1/4}}{(2\pi)^{1/2}n^{1/4}}\left[1+\left(\frac{5}{12}+\frac{(-1)^n}{4}\right)\frac{1}{n^{1/2}}+\cdots\right]$$

and the Hermite functions which satisfy

$$H_n(t)\sim 2^{(n+1)/2}(n/e)^{n/2}\exp(\tfrac{1}{2}t^2)\cos[(2n)^{1/2}t-\tfrac{1}{2}n\pi]$$

as $n\to\infty$ for fixed t.

The paper is expository in nature and few proofs are given. *W. K. Hayman* (London)

7296:

Suetin, P.K. Some asymptotic properties of polynomials. Dokl. Akad. Nauk. SSSR **129** (1959), 30–33. (Russian)

In his earlier paper [same Dokl. **114** (1957), 498–501; MR **19**, 852], the author gave asymptotic formulas for the generalized Faber polynomials and polynomials orthogonal over a contour. In this paper he discusses the problem of accuracy of the above results.

S. Kulik (Long Beach, Calif.)

7297:

Ferlan, Nives Maria; e Skof, Fulvia. Sulla permanenza della struttura lacunare attraverso il prolungamento analitico. Riv. Mat. Univ. Parma **8** (1957), 345–360.

Soit $f(z;\zeta)=\sum b_k^{(l)}(z-\zeta)^k$. Si $f(z,0)$ admet des lacunes $\{p_h,q_h\}$ ($b_k^{(0)}=0$ pour $p_h\leqq k\leqq q_h$, $h\geqq 1$), $f(z,\zeta)$, obtenu par le prolongement analytique le long d'une courbe, admet une structure lacunaire $\{p_h',q_h'\}$, c'est à dire:

$$\limsup|b_h^{(l)}|^{1/h}>\limsup|b_m^{(l)}|^{1/m}\ (p_h'\leqq m\leqq q_h',\ h\to\infty);$$

on détermine les intervalles $\{p_h',q_h'\}$ à partir de $\{p_h,q_h\}$.

S. Mandelbrojt (Paris)

7298:

de Possel, René. Sur les variétés à deux dimensions, les ensembles du type maximum et le prolongement des variétés conformes. C. R. Acad. Sci. Paris **249** (1959), 985–987.

Announcement of the following theorem which characterizes continuable Riemann surfaces in terms of a conformal universal covering with domain $\{|z|<1\}$. Let V denote a Riemann surface admitting a conformal universal covering φ with domain $\{|z|<1\}$. Then V is continuable if and only if either V has a planar boundary element or else the following condition is fulfilled: there exists a canonical fundamental polygon Π associated with φ such that $\Delta(\Pi)$, the union of the arcs δ_t of $\{|z|=1\}$ which have the same extremities as the arcs γ_t of Π in $\{|z|<1\}$ and do not contain points of fr Π, is not a set of maximum type (i.e., there exists a univalent conformal map f of $\{|z|<1\}$ onto a proper part of $\{|z|<1\}$ such that $|f|$ possesses the limit one at each point of $\Delta(\Pi)$). The notion of sets of maximum type was introduced by the

author in 1932 [same C. R. **194**, 42–44, 159–161, 585–587; and Thesis, J. École Polytech. **30** (1933), 1–98].

M. H. Heins (Urbana, Ill.)

7299:

Cornea, Aurel. Über eine Formel in der Extremisierungstheorie. Rev. Math. Pures Appl. **3** (1958), 431–436.

Let F be an open Riemann surface of hyperbolic type. Let Δ be an open set with relative boundary $\partial\Delta$ consisting of analytic curves which do not accumulate in F. Let U be a non-negative subharmonic function on F such that U is harmonic in Δ and 0 in $F-\Delta$. If U admits a non-negative harmonic majorant on F, then EU is defined to be the least harmonic majorant of U. Otherwise, $EU \equiv +\infty$. Concerning the extremization EU of U, the author obtains:

$$EU(p) = \frac{1}{2\pi}\int_{\partial\Delta} g(p,q)\,\frac{\partial U(q)}{\partial n}\,ds + U(p),$$

where $g(p, q)$ is the Green function of F and $\partial U/\partial n$ the inner normal derivative with respect to Δ. As an application of this formula, the author constructs a Riemann surface of parabolic type, with a single boundary component (in Kerékjártó-Stoïlow's sense) whose harmonic dimension [Heins, Ann. of Math. (2) **55** (1952), 296–317; MR **13**, 643] is infinite.

K. Noshiro (Nagoya)

7300:

Srivastav, R. P. On the entire functions and their derivatives represented by Dirichlet series. Ganita **9** (1958), 83–93.

Let $f(s)=\sum a_n e^{\lambda_n s}$, convergent for all s, with $\lambda_n^{-1}\log n \to 0$. Let $\mu(\sigma)=\max|a_n|e^{\lambda_n\sigma}$, the maximum term; $\nu(\sigma)$ the rank of the maximum term; $M(\sigma)=\max_t|f(\sigma+it)|$; let ρ denote the Ritt order and λ the lower Ritt order. The author shows that

$$\limsup\{\log\mu(\sigma)\}/\{\sigma\lambda_{\nu(\sigma)}\} \leqq 1-\lambda/\rho;$$

$$\limsup\{\log\mu(\sigma)\}/\{\lambda_{\nu(\sigma)}\log\lambda_{\nu(\sigma)}\} \leqq \lambda^{-1}-\rho^{-1};$$

$$\liminf\{\log\mu(\sigma)\}/\lambda_{\nu(\sigma)} \leqq \rho^{-1} \leqq \lambda^{-1} \leqq \limsup\{\log\mu(\sigma)\}/\lambda_{\nu(\sigma)}.$$

If $\mu_1(\sigma)$ refers to $f'(s)$, then

$$\limsup[\liminf]\,\sigma^{-1}\log\{\mu_1(\sigma)/\mu(\sigma)\} = \rho\,[\lambda].$$

If the condition $\lambda_n^{-1}\log n\to 0$ is replaced instead by $\sum(\lambda_{n+1}-\lambda_n)^{-1}<\infty$, then $\limsup \mu(\sigma)/M(\sigma)=1$.

R. P. Boas, Jr. (Evanston, Ill.)

7301:

Saginyan, A. L. On some inequalities and their applications in the theory of functions. Dokl. Akad. Nauk SSSR **129** (1959), 284–287. (Russian)

Let L denote an open Jordan arc that joins the origin to the unit circle $|z|=1$ and on which $|z|$ does not decrease as z moves toward the unit circle; suppose moreover that for each $r<1$ the portion of L lying in the disk $|z|\leq r$ is rectifiable. Let E denote a measurable set on L, and $E(r)$ the portion of E that lies in $|z|\leq r$. Suppose further that $\omega(r)$ is a non-negative bounded integrable function in $0\leq r<1$, satisfying the condition

$$\int_E \frac{\omega(|t|)\,d|t|}{1-|t|} < \infty.$$

For functions f meromorphic in $|z|<1$ the author gives an upper bound for $\log|f(z)|$ (too complicated to be reproduced here), expressed in terms of ω and the characteristic function T of f; and he concludes that if there exists a θ $(0<\theta<1)$ such that

$$\liminf_{r\to 1}\frac{\int_{E(r)}\omega(|t|)\log|f(t)|\,d|t|}{T[r+\theta(1-r)]} = -\infty,$$

then $f(z)\equiv 0$. He gives an analogous upper bound for $\log|f(z)|$ for the case where the set E is replaced by a set of positive two-dimensional measure. Finally, he obtains the analogues for the case where f is meromorphic in the entire plane.

G. Piranian (Ann Arbor, Mich.)

7302:

★**Hayman, W. K. Multivalent functions.** Cambridge Tracts in Mathematics and Mathematical Physics, No. 48. Cambridge University Press, Cambridge, 1958. viii+151 pp. 27s. 6d.

This book deals with that part of the theory of univalent and multivalent functions which is closest to the researches of the author. The majority of the results obtained throughout the book are of the classical inequality type. However, it is not to be inferred that only elementary methods are used in obtaining these results. For those with an introductory course in complex variables this book is self-contained, which is remarkable considering the length of the tract and the level reached.

The tract consists of six chapters. The first chapter covers much of the classical theory of univalent functions which can be obtained by elementary methods.

Chapter two and chapter three form a unit of study involving areally mean p-valent functions. A function $f(z)$ is called areally mean p-valent in an open set D if $f(z)$ is regular in D and the average (in the areal sense) number of roots in D of the equation $f(z)=w$ is not greater than p (a positive number) as w ranges over the disc $|w|<R$, $0<R<\infty$. The study of the rate of growth of $M(r;f)$ $(M(r;f)=\max_{|z|=r}|f(z)|$, $0<r<1)$, combining the work of Cartwright, Spencer, and the author, is given. This material is reworked, refined and unified to give a relatively lucid display of some very intricate techniques and results; in addition to this the works of Biernacki, Spencer and the author concerning the order of magnitude of coefficients of mean p-valent functions are given. In chapter four the author presents in detail the principles of circular and Steiner symmetrization, originally developed by Pólya and Szegö [*Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, N.J., 1951; MR **13**, 270], including various fundamentals such as Lipschitzian functions and detailed proofs of results usually heretofore considered as geometrically evident. The reader will find most of the applications of both circular and Steiner symmetrization in earlier works of the author [Tech. Rep. No. 11, Navy Contract N6-ori-106 Task Order 8, Stanford Univ., Calif., 1950; MR **12**, 401; J. Analyse Math. **1** (1951), 135–154; MR **13**, 545].

Suppose $f(z)$ is regular in D and $n(w)$ represents the number of roots of the equation $f(z)=w$ for z in D; if p is a positive number and $(2\pi)^{-1}\int_0^{2\pi} n(Re^{i\phi})\,d\phi \leq p$, $R>0$, then $f(z)$ is called circumferentially mean p-valent. This class of functions was first introduced by Biernacki [Bull.

Sci. Math. (2) **70** (1946), 45–51; MR **8**, 326]. In chapter five the author extends the results of the previous chapters to this class of functions, obtaining asymptotic expressions for the growth and coefficients of these functions. In particular a very general theorem concerning the asymptotic values of the coefficients is proven which, as a corollary, yields the Bieberbach conjecture for a fixed function $f(z)$ and for sufficiently large n.

In chapter six the author returns to the study of univalent functions. With the exception of one reference the Löwner theory set forth in this chapter employs no results from the tract other than in chapter one. The treatment of the Löwner theory is very much akin to that of Golusin [*Interior problems of the theory of schlicht functions*. Translated by T. C. Doyle et al. Office of Naval Research, Navy Dept., Washington, D.C., 1947; MR **8**, 575], but has been reworked in greater detail. The author gives the usual applications to the third coefficient problem, $\arg(f(z)/z)$, $\arg f'(z)$, radius of convexity and starlikeness and so on.

The length of the tract necessitates the omission of a great amount of material concerning univalent and multivalent functions. Since the variational methods of Schaeffer, Schiffer and Spencer and the theory of modules of Jenkins have appeared in book form the author is justified in omitting these subjects and referring the reader to them. However, even the reference to works of several other mathematicians on multivalent functions is noticeably absent. *W. C. Royster* (Lexington, Ky.)

7303:

Sakashita, Hideo. Some subclasses of the convex functions. Bull. Kyoto Gakugei Univ. Ser. B **14** (1959), 4–7.

Let $\mathfrak{R}$ denote the class of analytic functions F which are regular and univalent in the unit circle with $F(0)=0$, $F'(0)=1$, and which map the unit circle onto a convex domain. As F_1 varies through $\mathfrak{R}$, the functions F_2 defined by $F_2(z)=\int_0^z (F_1(z)/z)dz$ generate a class $\mathfrak{R}_2$ such that $\mathfrak{R}_2 \subset \mathfrak{R}$. Proceeding by induction, as F_m varies through the class $\mathfrak{R}_m$, the functions F_{m+1} defined by $F_{m+1}(z)=\int_0^z (F_m(z)/z)dz$ generate the class $\mathfrak{R}_{m+1}$ for which $\mathfrak{R}_{m+1} \subset \mathfrak{R}_m$ holds. The author shows how the distortion theorems and coefficient estimates for functions in $\mathfrak{R}$ can be sharpened for function in $\mathfrak{R}_m$, $m>1$. For example, if $F_m \in \mathfrak{R}_m$ and $F_m(z)=z+\sum_{\nu=2}^{\infty} a_\nu^m z^\nu$, then $|a_\nu^m| \leq 1/\nu^{m-1}$ $(\nu=2, 3, \cdots)$. If $F_m \in \mathfrak{R}_m$, then

$$\mathrm{Re}(F_m(z)/z) \geq \sum_{\nu=1}^{\infty} [(-1)^{\nu-1} r^{\nu-1}/\nu^{m-1}] > \sum_{\nu=1}^{\infty} (-1)^{\nu-1}/\nu^{m-1},$$

where $|z| \leq r < 1$. *G. Springer* (Lawrence, Kans.)

7304:

Epstein, Bernard; and Schoenberg, I. J. On a conjecture concerning schlicht functions. Bull. Amer. Math. Soc. **65** (1959), 273–275.

Let S be the class of all power series which converge and are univalent in the unit circle. It was conjectured that $\sum a_n z^n \in S$ and $\sum b_n z^n \in S$ implies $\sum n^{-1} a_n b_n z^n \in S$. This conjecture is disproved by the following counterexample. By a general result of Pólya and Schoenberg on de la Vallée-Poussin means [Pacific J. Math. **8** (1958), 295–334; MR **20** #7181], the polynomial $P(z)=15z+12z^2+3z^3$ lies in S. This polynomial is composed with the general power series in S which can be obtained from Loewner's differential equation. The composition series is again a polynomial $p(z)$ of degree 3. By proper choice of the Loewner parameter function one can achieve that $p'(z)$ vanishes in the unit circle. Thus, $p(z)$ cannot be univalent and a counterexample is constructed.
M. Schiffer (Stanford, Calif.)

7305:

Loewner, C.; and Netanyahu, E. On some compositions of Hadamard type in classes of analytic functions. Bull. Amer. Math. Soc. **65** (1959), 284–286.

We use the notations of the preceding review. It is shown that even the following conjecture is not true: "The composition series of two univalent power series has a non-vanishing derivative in $|z|<1$." This result disproves, a fortiori, the preceding conjecture. The counterexample is obtained as follows. All functions of the class S which map the unit circle onto the whole plane with a rectilinear slit and such that $z=1$ maps into ∞ have the representation $f_\varepsilon'(z)=(1+\varepsilon z)/(1-z)^3$ with constant ε, $|\varepsilon|=1$. For proper choice of ε and η the composition series of $f_\varepsilon(z)$ and $f_\eta(z)$ can be made to have a derivative which vanishes for $|z|<1$. *M. Schiffer* (Stanford, Calif.)

7306:

Shah, Tao-shing; and Chang, Kai-ming. Some inequalities in the theory of subordination. Acta Math. Sinica **8** (1958), 408–412. (Chinese. English summary)

Let $F(z)=\sum_{j=0}^{\infty} a_j z^j$ and $f(z)=\sum_{j=0}^{\infty} b_j z^j$ be regular in $|z|<1$. Let $F(z)$ conformally represent $|z|<1$ on a Riemann surface S_F. We say that $f(z)$ is subordinate to $F(z)$ if $a_0=b_0$ and $f(z)$ takes only values inside S_F. If S_f is a schlicht part of S_F, then $f(z)$ is said to be schlicht subordinate to $F(z)$.

If $f(z)$ is subordinate to $F(z)$, Rogosinski [Proc. London Math. Soc. (2) **48** (1943), 48–82; MR **5**, 36] established the inequality

$$|b_n| \leq \max(|a_1|, |a_2|, \cdots, |a_n|), \tag{1}$$

for $n=1$ and 2. In this paper Shah proves that

$$|b_3| \leq 2^{1/2} \max(|a_1|, |a_2|, |a_3|)$$

and that the constant $2^{1/2}$ is the best possible (Theorem 1). Chang points out that (1) holds for $n=3$, if $f(z)$ is schlicht subordinate to $F(z)$ (Theorem 2). Finally Chang proves that if $f(z)$ is subordinate to $F(z)$, $f(0)=F(0)=0$, and $F(z)$ is schlicht in $|z|<1$, then the map S_f covers the circle $|w| \leq \frac{1}{2}|b_1| \cdot |b_1/a_1|$, but not any larger circle (Theorem 3).

In proving theorem 3, the Schwarz lemma is used. In proving theorems 1 and 2, the following lemmas are used: Let the class of functions $w(z)$ regular in $|z|<1$, $w(0)=0$, $|w(z)|<1$, be denoted by B. (1) If $w(z) \in B$, $w(z)=c_1 z+c_2 z^2+\cdots$, then

$$|c_1| \leq 1, \quad |c_2(1-|c_1|^2)+\bar{c}_1 c_2^2| \leq (1-|c_1|^2)^2-|c_2|^2$$

and conversely; (2) if $w(z) \in B$ and is schlicht then $|c_2| \leq 2|c_1|(1-|c_1|)$; (3) under the same condition of lemma (2) and for fixed $|c_1|$ and α, $0 \leq \alpha < 1$, we have

$$|c_1 c_2 - \alpha c_2^2| \leq |c_1|^2(1-|c_1|^2)$$

when $|c_1| \geq \exp(-1/(1-\alpha))$.
C. Y. Wang (Minneapolis, Minn.)

7307:

Chou, Hsin-ti. Concerning the properties of bounded regular function in the neighborhood of the boundary. Acta Math. Sinica **8** (1958), 396–407. (Chinese. English summary)

The author proves four main theorems together with eleven corollaries under theorem 3. The first theorem is said to be an extension of the Tauberian lemma by Eggleston [Proc. London Math. Soc. (3) **3** (1951), 28–45; MR **13**, 22]. The four theorems may be stated as follows. Let $f(z)$ be a bounded regular function with a finite number of zeros in a domain D, and z_0 be a point on the frontier of D. z_0 may be the point at infinity, but in any case the frontier of D must contain at least one finite point. Let $S \subset D$ be a set of points, z_0 be one of its limit points; for $z \in D$, let $d(z)$ be the distance of z from the frontier of D. (1) If $\lim_{S \ni z \to z_0} f(z) = 0$, then

$$\lim_{S \ni z \to z_0} f^{(P)}(z)(d(z))^P = 0 \quad (P = 1, 2, \cdots).$$

(2) Let G_x be a subset of D which consists of (i) all the points of S, (ii) all the points z of D which satisfy $|z - z_1| < x d(z_1)$ for some z_1 of S, where x is a fixed number, $0 < x < 1$. If $\lim_{S \ni z \to z_0} f(z) = 0$, then $\lim_{G_x \ni z \to z_0} f(z) = 0$ uniformly in G_x, as $|z - z_0| \to 0$ (or as $|z| \to +\infty$ if $z_0 = \infty$). (3) Suppose $f(z)$ is a bounded regular function in D, and for any complex number α the number of roots of $f(z) - \alpha = 0$ is finite. Then

$$\lim_{D \ni z \to z^*} f^{(P)}(z)(d(z))^P = 0 \quad (P = 1, 2, \cdots),$$

where z^* is any point on the frontier of D. (4) Let $f(z)$ be a bounded regular function in D, and $f(z) \neq 0$ for $z \in D$. If the frontier of D contains at least one finite point, then

$$|f^{(k)}(z)|(d(z))^k \leq 2(k!)^2 \left| |f(z)| \log \frac{m}{|f(z)|} \right|,$$

$z \in D$, $k = 1, 2, \cdots$, where $m = \max_{z \in D} |f(z)|$.

In proving the first theorem the author uses the lemma: If $f(z)$ is a bounded regular function in $|z| < 1$ such that

$$(2\pi)^{-1} \int_0^{2\pi} |\mathrm{Re}\, f(re^{i\theta})| d\theta \leq M \quad (0 \leq r < 1),$$

then

$$|a_k| = |f^{(k)}(0)/k!| \leq 2M \quad (k = 1, 2, \cdots).$$

He defines $Z = (z^* - z)/d(z)$, $z \in S$, z^* variable in D, and

$$F(Z) = f[d(z)Z + z]/\mathrm{Re}\, f(z), \quad z \in S, \quad |Z| < 1.$$

In proving theorem 2 he uses the Taylor's expansion of $f(z)$ and the estimate from theorem 1. In proving theorem 3 he defines $F(z) = 1/\log(M/f(z))$ and makes use of the Cauchy inequality. In proving theorem 4 he defines $F(z) = \log(m/f(z))$, obtains a general form of $f^{(k)}(z)$, then obtains the proper bound by the arguments related to polynomials.

The abstract in English is not arranged in the same order as his paper. Corollaries 1 and 2 correspond to theorems 2 and 3 respectively. Corollaries 3–9 correspond to corollaries 8, 1, 2, 3, 5, 4, 6 under theorem 3 respectively. *C. Y. Wang* (Minneapolis, Minn.)

7308:

Tumarkin, G. C. Sequences of Blaschke products. Dokl. Akad. Nauk SSSR **129** (1959), 40–43. (Russian)

The author announces, without proof, numerous results concerning properties of the limit function $\mathfrak{B}(z)$ of a sequence of Blaschke products

$$b_k(z) = \prod_{j=1}^{\infty} \frac{\alpha_{kj} - z}{1 - \bar{\alpha}_{kj} z} \frac{|\alpha_{kj}|}{\alpha_{kj}} \quad (k = 1, 2, \cdots),$$

which converges uniformly in $|z| < 1$. The criteria characterizing these properties of $\mathfrak{B}(z)$ are stated solely in terms of the zeros $\{\alpha_{kj}\}$ of the functions $\{b_k(z)\}$. Thus, necessary and sufficient conditions are given for: (I) $\mathfrak{B}(z) \equiv 0$ in $|z| < 1$; (II) $\mathfrak{B}(z)$ itself a Blaschke product; (III) $\mathfrak{B}(z) \neq 0$ in $|z| < 1$; (IV) given a set E on $|z| = 1$, $|\mathfrak{B}(e^{i\theta})| = 1$ almost everywhere on E. By way of example, the condition for II reads: (1) In every disk $|z| < r$, $0 < r < 1$, the number of zeros of $b_k(z)$ is uniformly bounded; (2) to each $\varepsilon > 0$, there exists an $R(\varepsilon)$, $0 < R < 1$, such that for every $k = 1, 2, \cdots$, $\sum (1 - |\alpha_{kj}|) < \varepsilon$, where the summation is extended over all zeros of $b_k(z)$ whose moduli are greater than R.

Among other interesting results, we single out the following: If $\tilde{\mathfrak{B}}(z)$ denotes an arbitrary holomorphic function in $|z| < 1$ for which $|\mathfrak{B}(z)| \leq |\tilde{\mathfrak{B}}(z)| \leq 1$, then by removing suitable factors from each $b_k(z)$, one can obtain a sequence of new products $\{\tilde{b}_k(z)\}$ which converges uniformly in $|z| < 1$ to $e^{i\gamma}\tilde{\mathfrak{B}}(z)$, where γ is a real number.

W. Seidel (Notre Dame, Ind.)

7309:

Effertz, F. H.; and Breuer, K. H. Ein Algorithmus und ein Klassifikationsprinzip für Funktionen mit nichtnegativem Realteil. Math. Ann. **138** (1959), 335–341.

$F(\lambda)$ is called a non-negative function if, for $\mathrm{Re}(\lambda) > 0$, $F(\lambda)$ is analytic and $\mathrm{Re}[F(\lambda)]$ is non-negative. The following generalization of a theorem of Fialkow and Gerst [J. Math. and Phys. **34** (1955), 160–168; MR **17**, 435] is proved. Let $F(\lambda)$ be a non-negative function and λ_i $(i = 1, 2, \cdots, m)$ a sequence of numbers with $\mathrm{Re}(\lambda_i) > 0$ and $F(\lambda_i) = s$ $(i = 1, 2, \cdots, m)$. Suppose $F(\lambda)$ is not a function of the form

$$[s\tilde{g}(\lambda) + \bar{s}g(-\lambda)]/[\tilde{g}(\lambda) - sg(-\lambda)]$$

with $|s| = 1$ and $g(\lambda)$ the Hurwitz polynomial $g(\lambda) = \prod_{i=1}^{m} (\lambda + \lambda_i)$. Then

$$F_1(\lambda) = [Kg(-\lambda)\{\bar{s} + F(\lambda)\} - \bar{K}\tilde{g}(\lambda)\{s - F(\lambda)\}]/[g(-\lambda)\{\bar{s} + F(\lambda)\} + \tilde{g}(\lambda)\{s - F(\lambda)\}]$$

with $\mathrm{Re}(K) > 0$ is a non-negative function. This algorithm for the construction of a sequence of non-negative functions from a given one is used as a criterion for classifying such functions. *A. Fialkow* (Brooklyn, N.Y.)

7310:

Nishimiya, Han. On coefficient-regions of Laurent series with positive real part. Kōdai Math. Sem. Rep. **11** (1959), 25–39.

Let R_0 be the class of analytic functions $1 + c_1 z + c_2 z^2 + \cdots$ with a positive real part in $|z| < 1$. According to a classical result of Carathéodory, the coefficient space $(c_1, c_2, \cdots, c_n)$ of R_0 is the convex hull of the points $(2e^{it}, 2e^{2it}, \cdots, 2e^{nit})$, $0 \leq t < 2\pi$. The extremal functions corresponding to these boundary points are the functions $(1 + e^{it}z)(1 - e^{it}z)^{-1}$ which map $|z| < 1$ onto the right half-plane. The author generalizes this result to the class R_ϱ of Laurent series $1 + \sum_{n=-\infty}^{\infty}{}' c_n z^n$ with a positive real part

in $q<|z|<1$. He shows that the extremal functions spanning the convex coefficient space $(c_{-m}, \cdots, c_n)$ of R_q are elliptic functions which map the ring $q<|z|<1$ onto the right half-plane with a slit along the real axis. As an application of this result he obtains the sharp inequality $|c_n| \leq 2|1-q^n|$. The author was evidently not aware of the fact that this inequality appears in an earlier paper of the reviewer [J. London Math. Soc. **25** (1950), 19–26; MR **11**, 435]. *Z. Nehari* (Pittsburgh, Pa.)

7311:

Walsh, J. L.; and Russell, H. G. Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions. Trans. Amer. Math. Soc. **92** (1959), 355–370.

Soit, dans le plan complexe des z, C un ensemble de λ courbes simples analytiques C_j $(j=1, 2, \cdots, \lambda)$, extérieures l'une à l'autre, $\bar{C}_j$ la fermeture de l'intérieur de C_j; soit γ le cercle unité du plan des $w=re^{i\theta}$, $z=\chi_j(w)$ une fonction faisant la représentation conforme de l'intérieur de γ sur l'intérieur de C_j; soit $w=\varphi(z)$ une fonction faisant la représentation conforme de l'extérieur de C sur $|w|>1$, avec $\varphi(\infty)=\infty$ (φ n'est pas biunivoque pour $\lambda>1$); par C_ρ on désigne l'image de $|w|=\rho(>1)$. Une fonction $f(w)$ holomorphe à l'intérieur de γ est dite de classe H_p si: $\{\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\}^{1/p}$ est bornée pour $0<r<1$ $(p \geq 1)$; $f(z)$ holomorphe dans C (c'est-à-dire holomorphe à l'intérieur de tous les C_j) est dite de classe H_p si $f[\chi_j(w)] \in H_p$, quel que soit j.

Soit s une abscisse curviligne sur C_j, supposons $f(z)$ holomorphe dans C soit telle que $f^{(k)}(z) \in H_p$, et soit $f^{(k)}(z)=F_j(s)$, posons:

$$\omega_{pj}(\delta, f^{(k)}) = \max_{0<|h|\leq\delta} \left\{\int_{C_j} |F_j(s+h)-F_j(s)|^p ds\right\}^{1/p},$$

$$\omega_p(\delta, f^{(k)}) = \max_j \omega_{pj},$$

$$\omega_{pj}^*(\delta, f^{(k)}) = \max_{0<|h|\leq\delta} \left\{\int_{C_j} |F_j(s+h)+F_j(s-h) - 2F(s)|^p ds\right\}^{1/p}, \quad \omega_p^*(\delta, f^{(k)}) = \max_j \omega_{pj}^*;$$

on dit qu $f \in H(k, \alpha, p)$ si $\omega_p(\delta, f^{(k)}) \leq M\delta^\alpha$ (avec $0<\alpha<1$), et que $f(z) \in Z(k, p)$ si $\omega_p^*(\delta, f^{(k)}) \leq M\delta$, où M désigne, comme dans tout ce qui suit, une constante positive, indépendente de n, z et w.

On a alors les principaux résultats suivants: (1) Si $f \in H(k, \alpha, p)$ (avec $0<\alpha<1$), ou $f \in Z(k, p)$ (ce qui correspond à $\alpha=1$), il existe un polynome $\pi_n(z)$ de degré $\leq n$, tel que:

$$\int_C |f(z)-\pi_n(z)|^p |dz| \leq M/n^{(k+\alpha)p}.$$

(2) Si $F(z)$, holomorphe dans C, satisfait à une inégalité de cette forme (F au lieu de f), alors elle tend presque-partout sur C vers une fonction $f \in H(k, \alpha, p)$, ou $f \in Z(k, p)$, suivant que $0<\alpha<1$, ou que $\alpha=1$. (Généralisations de résultats connus dans le domaine réel: Bernstein, Jackson, de la Vallée Poussin, Montel, Zygmund.) (3) Si C_ρ, comme C, se compose d'un nombre fini de courbes extérieures l'une à l'autre, prenant à présent k entier, mais non nécessairement ≥ 0, et si $f \in H(k, \alpha, p)$, ou $Z(k, p)$ sur C_ρ, alors il existe des polynomes π_n tels que: $|f-\pi_n| \leq M/n^{k+\alpha} \cdot \rho^n$, pour $z \in \bar{C}$. (4) Si C se compose d'une seule courbe, f étant définie sur C, s'il existe une suite de polynomes π_n tels que:

$$\int_C |f-\pi_n|^p |dz| < M/n^{(k+\alpha+1)p} \cdot \rho^{np} \quad (0<\alpha\leq 1, p>1),$$

alors f peut être convenablement définie sur C pour être prolongée analytiquement dans C_ρ, et c'est une fonction de classe $H(k, \alpha, \infty)$ sur C_ρ (pour $0<\alpha<1$), ou de classe $Z(k, \infty)$ (pour $\alpha=1$). Si $k \geq 0$, on a: $|f-\pi_n| < M/n^{k+\alpha}$, pour $z \in \bar{C}_\rho$. (Extensions de résultats de Hardy et Littlewood.) Avec des modifications convenables, on peut étendre certains de ces résultats au cas de l'approximation par des suites de fonctions bornées convenables.

J. Favard (Paris)

7312:

Erohin, V. On the best approximation of analytic functions by rational fractions with free poles. Dokl. Akad. Nauk SSSR **128** (1959), 29–32. (Russian)

Let A_R consist of all functions analytic in the circle $|z|<R$, $R>1$, and let $\mathfrak{R}_n$ be the set of all rational functions R_n of order $\leq n$ (i.e., of quotients of polynomials of degree $\leq n$) which have no poles in $|z| \leq 1$. Comparing the degrees of approximation of functions $f \in A_R$ by the R_n and by the polynomials of degree $\leq n$, the author finds that for the "worst" functions of A_R the former is asymptotically not smaller than the latter.

Let $\|f\|$, $\|f\|_p$, $\|f\|_p'$ $(p \geq 1)$ denote the norms $\sup_{|z|<1}|f(z)|$, $((2\pi)^{-1}\int_{|z|=1} |f(z)|^p |dz|)^{1/p}$, $(\pi^{-1}\iint_{|z|<1}|f(z)|^p |dz|^2)^{1/p}$; and let $\tau_R=\sup_{f\in A_R}\tau(f)$, $\tau(f)=\lim\sup_{n\to\infty} \varepsilon_n(f)^{1/n}$, $\varepsilon_n(f)=\inf_{R_n\in\mathfrak{R}_n}\|f-R_n\|$; $\tau_R^*=\lim\sup_{n\to\infty}(\varepsilon_{n,R})^{1/n}$, $\varepsilon_{n,R}=\sup\varepsilon_n(f)$, where the supremum is taken for all $f \in A_R$ with $\|f\|_R \leq 1$ and $\|f\|_R$ is the uniform norm in the circle $|z|<R$. The quantities $\tau_{p,R}$, $\tau_{p,R}^*$ and $\tau_{p,R}'$, $\tau_{p,R}^{*'}$ are defined by replacing in these relations the norm $\|f\|$ by $\|f\|_p$ and by $\|f\|_p'$. The main result is that all these τ are equal to R^{-1}, which is the corresponding value for the polynomial approximation. For $\|f\|_2<+\infty$ or $\|f\|_2'<+\infty$ the rational functions of best approximation are contained in the formula

$$R_n(z) = f(z)-(2\pi i)^{-1}zB_n(z)\int_{|t|=\rho} f(t)(t-z)^{-1}t^{-1}B_n(t)^{-1}dt,$$

$(|z|<\rho<1)$, where $B_n(z)$ is the Blaschke product $\prod_{k=1}^n (z-a_k)(1-\bar{a}_k z)^{-1}$, and the a_k assume values $|a_k|<\rho$. Possibilities of the determination of the a_k are outlined. Another formula for $p=2$, $R \geq 2$ is

$$\tau_2(f) \leq \max[R^{-1}\lim\sup_{n\to\infty} h_n^{1/n}, 2(R^2+1)^{-1}],$$

where $h_n=\min H_n(z_1, \cdots, z_n)$, $\sum_1^n |z_k|^2 \leq 1$, H_n^2 is the quadratic form $\sum_{j=1}^n |g_{k+n}-\sum_{j=1}^n g_{k+j-1}z_j|^2$, and the g_ν are given by $f(z)=\sum_{\nu=0}^\infty R^{-\nu}g_\nu z^\nu \in A_R$. If $f(z)=\sum R^{-\lambda_m}z^{\lambda_m}$ and the integers λ_m increase sufficiently rapidly, then $\tau_2(f)=R^{-1}$. *G. G. Lorentz* (Syracuse, N.Y.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 7357, 7533, 7740.

7313:

Bavrin, I. I. Estimates in the theory of analytic functions of two complex variables. Dokl. Akad. Nauk SSSR **126** (1959), 919–922. (Russian)

In the space of two complex variables z, w, let D be a doubly-circular domain whose boundary is given by $|w|=r_1(\tau)$, $|z|=r_2(\tau)$, $0\leqq\tau\leqq 1$, where $r_1(0)=0$, $0<r_1'(\tau)\leqq r_1(\tau)/\tau$, $r_1(1)<\infty$ and $\ln r_2(\tau)=-\int \tau/(1-\tau)\,d\ln r_1(\tau)$.

With any point (z, w) of D, the author associates the "absolute value"

$$a=\max_{0<\tau<1}\left[\tau\frac{|w|}{r_1(\tau)}+(1-\tau)\frac{|z|}{r_2(\tau)}\right],$$

and he asserts that this quantity verifies many familiar estimates from the theory of "schlicht" functions in one variable. For instance, if $\operatorname{Re} F(z, w)>0$ in D, then

$$|\operatorname{Im} F(z, w)|\leqq F(0, 0)\frac{2a}{1-a^2},$$

and if $F(0, 0)=0$, $|\operatorname{Re} F(z, w)|<1$, then

$$|F(z, w)|\leqq\frac{2}{\pi}\ln\frac{1+a}{1-a}.$$

S. Bochner (Princeton, N.J.)

7314:

Litvinčuk, G. S. On completeness of some systems of analytic functions of two variables. Dokl. Akad. Nauk SSSR **128** (1959), 37–40. (Russian)

A system S of holomorphic functions is called complete in a domain D if every function holomorphic in D is the uniform limit on compact subsets of linear combinations of functions in S. Given a function $f(z, w)$ holomorphic in a bicylinder, the author indicates various ways of constructing complete systems from it. E.g., under certain assumptions the system of all the iterated integrals of f is complete; or, under some restrictions on the derivatives of f the system of all $f(z^m, w^n)$ $(m, n=1, 2, \cdots)$ is complete. The theorems are too numerous and lengthy to be stated here in detail.

A. Korányi (Cambridge, Mass.)

7315:

Kajiwara, Joji. Some results on the equivalence of complex-analytic fibre bundles. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 37–48.

Der Verfasser betrachtet komplex analytische Faserbündel über holomorph konvexen komplexen Mannigfaltigkeiten M. Er bezeichnet mit D ein Teilgebiet von M und mit D^* die pseudokonvexe Hülle von D. (Dieser Begriff wird in der Arbeit definiert.) Er zeigt unter der Voraussetzung, daß die Strukturgruppe holomorph vollständig ist, daß 2 komplex analytische Faserbündel, die über D äquivalent sind, auch über D^* es sind. Die Aussagen, die der Verfasser in der Arbeit von H. Grauert, Ann. of Math. (2) **68** (1958), 460–472 [MR **20** #5299], gefunden hat, stehen nicht dort. Sie werden in einer Arbeit von T. Docquier und H. Grauert über *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten* veröffentlicht werden, die in Kürze in den Math. Ann. erscheint. *H. Grauert* (Göttingen)

SPECIAL FUNCTIONS

See also 7296, 7386, 7409.

7316:

Saermark, K. A note on addition theorems for Mathieu functions. Z. Angew. Math. Phys. **10** (1959), 426–428. (German summary)

Consider two elliptic cylinder coordinate systems A, B, with cylinder axes parallel; A with centre at the origin and foci on the x-axis, B with arbitrary centre and foci. The Meixner and Schäfke addition theorem for Mathieu functions [see Meixner and Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen*, Springer-Verlag, Berlin, 1954; MR **16**, 586] is, effectively, the expansion of an elliptic cylinder wave in the B-system as the sum of elliptic cylinder waves in the A-system, valid in an outer region excluding the foci of both systems. In this paper the author gives a similar expansion valid in an inner region containing the foci of the A-system but not those of the B-system. *F. M. Arscott* (London)

7317:

Carlitz, L. Multiplication formulas for products of Bernoulli and Euler polynomials. Pacific J. Math. **9** (1959), 661–666.

Put

$$te^{xt}(e^t-1)^{-1}=\sum_{n=0}^{\infty}B_n(x)t^n/n!$$

and let $\bar{B}_m(x)$ be the Bernoulli function defined by

$$\bar{B}_m(x)=B_m(x)\quad(0\leqq x<1),$$
$$\bar{B}_m(x+1)=\bar{B}_m(x)\quad(x\geqq 0).$$

The author proves three theorems of which the following is theorem 1. Let $n\geqq 1$; $m_1, \cdots, m_n\geqq 1$; $a_1, a_2, \cdots, a_n$ positive integers that are relatively prime in pairs; $A=a_1a_2\cdots a_n$. Then

$$\sum_{r=0}^{kA-1}\bar{B}_{m_1}\left(x_1+\frac{r}{a_1k}\right)\bar{B}_{m_2}\left(x_2+\frac{r}{a_2k}\right)\cdots\bar{B}_{m_n}\left(x_n+\frac{r}{a_nk}\right)=$$
$$C\sum_{r=0}^{k-1}\bar{B}_{m_1}\left(a_1x_1+\frac{r}{k}\right)\bar{B}_{m_2}\left(a_2x_2+\frac{r}{k}\right)\cdots\bar{B}_{m_n}\left(a_nx_n+\frac{r}{k}\right),$$

where $C=a_1^{1-m_1}a_2^{1-m_2}\cdots a_n^{1-m_n}$. Theorem 2 is analogous to theorem 1 but involves the Euler function $\bar{E}_m(x)$ in place of the Bernoulli function $\bar{B}_m(x)$. Theorem 3 is an extension of theorem 2 obtained by introducing the "Eulerian" polynomial $\phi_m(x, \rho)$ defined by

$$\frac{1-\rho}{1-\rho e^t}e^{xt}=\sum_{m=0}^{\infty}\phi_m(x, \rho)\frac{t^m}{m!}\quad(\rho\neq 1),$$

where, in particular, $\phi_m(x, -1)$ reduces to the Euler polynomial $E_m(x)$. *A. L. Whiteman* (Princeton, N.J.)

7318:

Singh, V. N. A note on the computation of Alder's polynomials. Pacific J. Math. **9** (1959), 271–275.

L'auteur poursuit ses recherches sur les généralisations des identités de Ramanujan, de la forme

$$\prod_{n=1}^{\infty}\frac{[1-x^{(2M+1)n-M}][1-x^{(2M+1)n-M-1}][1-x^{(2M+1)n}]}{1-x^n}=\sum_{t=0}^{\infty}\frac{G_{M,t}(x)}{(x;t)}$$

et donne des formules permettant le calcul de $G_{M,t}(x)$.

R. Campbell (Caen)

ORDINARY DIFFERENTIAL EQUATIONS

See also 7411, 7413, 7497, 7498, 7660, 7715.

7319:

Brauer, Fred. Some results on uniqueness and successive approximations. Canad. J. Math. **11** (1959), 527–533.

Let f be a continuous n-dimensional vector-valued function defined for real t, $0<t<a$, and for $|x|\leq b$, and satisfying there

$$|f(t, x_1)-f(t, x_2)| \leq \psi_i(t, |x_1-x_2|) \quad (i = 1, 2).$$

Here ψ_1, ψ_2 are two continuous non-negative functions defined for $0<t<a$, $r\geq 0$, which are monotone non-decreasing in r for each fixed t. Suppose A and B are functions on $0\leq t<a$ with $A(0)=B(0)=0$, such that $A(t)/B(t)\to 0$ $(t\to 0)$. Suppose that all solutions u of $u'=\psi_1(t, u)$, $u(0)=0$, satisfy $u(t)\leq A(t)$ on $0\leq t<a$, and that the only solution v of $v'=\psi_2(t, v)$ on $0\leq t<a$ such that $v(t)/B(t)\to 0$ $(t\to 0)$ is the trivial solution. Under these assumptions it is shown that there is at most one solution of $x'=f(t, x)$, $x(0)=0$, on $0\leq t<a$. If, in addition, $|f(t, x)|\leq M$ for $0<t<a$, $|x|\leq b$, the successive approximations converge on $0\leq t<\min(a, b/M)$ to this unique solution there. These results generalize a number of previous results concerning the same questions.

E. A. Coddington (Los Angeles, Calif.)

7320:

de Losada y Puga, Cristóbal. On the properties of the solutions of second order homogeneous linear differential equations with constant coefficients. Actas Acad. Nac. Ci. Lima **22** (1959), no. 1/2, 1–16. (Spanish)

A study of the behaviour of the integral curves of second order homogeneous linear differential equations with constant coefficients $y''+2py'+qy=0$, for values of p and q satisfying each of the conditions $p^2-q\gtreqless 0$. Intercepts, extrema, inflection points and the point of intersection of two curves of the family are determined.

M. O. Gonzalez (Havana)

7321:

Denjoy, Arnaud. Les équations différentielles périodiques. C. R. Acad. Sci. Paris **249** (1959), 590–591.

The complicated construction of the previous note [same C. R. **248** (1959), 1253–1258; MR **21** #5042] is examined further. A property called normal ergodicity is said to hold. Several questions concerning skeins and cycles are asked.

L. W. Green (Minneapolis, Minn.)

7322:

Hille, Einar. An application of Prüfer's method to a singular boundary value problem. Math. Z. **72** (1959/60), 95–106.

It is known that under certain conditions the operator $b(x)d^2/dx^2$ subject to the boundary condition $y(0)=0$ has pure point spectrum $0<\lambda_0<\lambda_1<\lambda_2<\cdots$, $\lambda_n\to\infty$ on $C[0, \infty]$. In this paper the author uses a modification of the polar coordinate approach to prove two theorems on the order of growth of λ_n as a function of n. Theorem 1: If $b(x)=(x+1)^2\omega(x)$, where (i) $\omega(x)>0$, (ii) $\omega(x)$ is continuously differentiable and $\omega'(x)>0$ for large x, and (iii) $J=\int_0^\infty |b(x)|^{-1/2}dx<\infty$; then $\lambda_n=(\pi n/J)^2[1+o(1)]$ as $n\to\infty$. Theorem 2: Suppose that (i) and (ii) hold, but that instead of (iii) it is required that (iv) for every $\varepsilon>0$ and $C>1$ there is an $x(\varepsilon, C)$ such that $\omega(x)<\omega(y)<(1+\varepsilon)\omega(x)$ when $x(\varepsilon, C)\leq x<y<Cx$, (v) $\int_0^\infty x[b(x)]^{-1}dx<\infty$, and (vi) there is an integer $p\geq 1$ such that

$$\limsup_x [\inf] [\omega(x)]^{1/2}[L_p(x)]^{-1}\int_0^x (1+s)^{-1}[\omega(s)]^{-1/2}ds = \mu_p[\beta_p],$$

where $L_p(x)=\log x \log_2 x \cdots \log_p x$, $\log_p x$ is the p times iterated logarithm, and $1\leq\beta_p\leq\mu_p<\infty$ except for $p=1$ when we require $1<\beta_1$. Then $\lambda_n=\rho_n^2\omega[L_p^{-1}(\sigma_n n)]$ where $\rho_n>\frac{1}{2}$ and tends to $\frac{1}{2}$ as $n\to\infty$, and L_p^{-1} is the inverse function of L_p. The limits of indeterminacy of σ_n are $2\pi\mu_p^{-1}\leq\liminf\sigma_n\leq\limsup\sigma_n\leq 2\pi\beta_p^{-1}$ for $p>1$. For $p=1$, μ_p^{-1} is replaced by $(\mu_1+1)^{-1}$ and β_p^{-1} by $(\beta_1-1)^{-1}$.

R. R. D. Kemp (Kingston, Ont.)

7323:

Martin, Allan D. An inverse Sturm-Liouville problem. Duke Math. J. **26** (1959), 455–465.

With $r(x)$ and $p(x)$ fixed real-valued and continuous functions over real $x\in(a, b)$, and with $r(x)>0$ there, consider real-valued solutions u of

$$\frac{d}{dx}\{r(x)u'(x)\}+p(x)u(x) = 0$$

there, with $u'(x)$ existing and continuous there, which satisfy $u(c)\cos\theta+r(c)u'(c)\sin\theta=0$ for some $c\in(a, b)$ and $\theta\in(0, \pi)$. Define, where existent in (a, b), $f(c, \theta)$ as the first zero of such u to the right of c for such c, θ; define $f(c, 0)=c$. Calling f the focal function for r and p, the author proves that the focal function for r and p determines r and p uniquely, and gives sufficient criteria that a given function f be the focal function for some such r, p.

F. H. Brownell (Seattle, Wash.)

7324:

Opial, Z. Sur l'allure asymptotique des solutions de l'équation différentielle $u''+a(t)u'+b(t)u=0$. Ann. Polon. Math. **6** (1959), 181–200.

Es sei, für $t\geq t_0$, (1) $0\leq c\leq b(t)\leq C$, $2\sqrt{C}\leq a(t)$, wo c und C zwei positive Konstanten sind. Der Verfasser nennt die Gleichung (2) $u''+a(t)u'+b(t)u=0$ vom Typus (Z), wenn jede Lösung bei $t\to\infty$ gegen Null konvergiert und vom Typus (M), wenn zwei linear unabhängige Lösungen existieren, von denen eine gegen Null und die andere gegen einen von Null verschiedenen Grenzwert konvergiert. Unter der Voraussetzungen (1) ist die Gleichung (2) entweder vom Typus (Z) oder vom Typus (M). Der Verfasser hat unlängst [Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 847–853; MR **19**, 1051] Bedingungen angegeben, welche hinreichend und manche auch notwendig sind, dass die Gleichung (2) vom Typus (Z) oder (M) sei. In diesem Artikel führt er sie wieder ein und beweist einige neue Resultate. Wir nennen zwei von ihnen: 1. Wenn, für $t\geq t_0$, $2\leq a_1(t)\leq a(t)$ gilt und die Gleichung $u''+a(t)u'+u=0$ vom Typus (M) ist, so ist die Gleichung $u''+a_1(t)u'+u=0$ vom gleichen Typus. 2. Die Bedingung

$$\int_0^\infty\left(\frac{1}{A(s)}\int_0^s A(r)dr\right)ds = \infty \quad \left(A(t)=\exp\left(\int_0^t a(s)ds\right)\right),$$

welche hinreichend und notwendig ist, damit (2) vom Typus (Z) sei, kann man nicht durch die einfachere $\int_0^\infty ds/a(s)=\infty$ ersetzen.

M. Zlámal (Brno)

7325:

Opial, Z. Sur les valeurs asymptotiques des intégrales des équations différentielles linéaires du second ordre. Ann. Polon. Math. **6** (1959), 201–210.

Der Verfasser beschäftigt sich mit der Gleichung $u''-a(t)u'+b(t)u=0$ hauptsächlich unter den Voraussetzungen (1) aus der vorhergehenden Rezension. Es sind nur zwei Fälle möglich: Entweder konvergiert jede Lösung bei $t\to\infty$ gegen Unendlich oder es existieren zwei linear unabhängige Lösungen, von denen eine gegen Unendlich und die andere gegen einen endlichen, von Null verschiedenen Grenzwert konvergiert. Der Verfasser führt Bedingungen ein, welche ähnlich denen von der vorhergehenden Rezension sind und welche hinreichend und zum Teil auch notwendig sind, dass der erste oder der zweite Fall eintrete. *M. Zlámal* (Brno)

7326:

Olech, C. Asymptotic behaviour of the solutions of second order differential equation. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 319–326. (Russian summary, unbound insert)

Let $a(t)$, $b(t)$ be real and continuous for all t and let $\lambda^2+a(t)\lambda+b(t)=0$ have real roots $\lambda_1(t)\leq\lambda_2(t)$. It is shown that if there are constants $\alpha,\beta,\gamma,\delta$ such that $-\infty\leq\alpha\leq\lambda_1(t)\leq\beta<\gamma\leq\lambda_2(t)\leq\delta\leq\infty$, then (1) $x''+a(t)x'+b(t)x=0$ has non-vanishing solutions x_1, x_2 satisfying $\alpha\leq x_1'/x_1\leq\beta$ and $\gamma\leq x_2'/x_2\leq\delta$ for all t. A corresponding result holds if β,γ are (finite-valued) non-increasing and α,δ are (finite-valued) non-decreasing functions of t.

P. Hartman (Baltimore, Md.)

7327:

Fort, Tomlinson. A problem in linear differential equations. Proc. Amer. Math. Soc. **10** (1959), 300–303.

Let $u''+(1+\lambda g(x))u=0$ with $\int_0^\infty|g|dx<\infty$ and let $u_1(x)$ be the solution of this equation with the initial data $u_1(0)=0$, $u_1'(0)=1$. It is known that $u_1(x)\sim r(\lambda)\sin(x+\theta(\lambda))$. Following a suggestion of Bellman, the author studies the analytic properties of $r(\lambda)$ and $\theta(\lambda)$ in dependence on the complex parameter λ. He shows that $r(\lambda)^2$ is an entire function and $\cos\theta(\lambda)$ is analytic and regular except at the points where $r(\lambda)=0$. The theorem is proved by considering first an analogous problem for a difference equation and then passing to the limit.

M. Schiffer (Stanford, Calif.)

7328:

Kowalski, Z. Generalized characteristic directions for a system of differential equations. Ann. Polon. Math. **6** (1959), 269–280.

This paper is concerned with the existence of solutions of the system (1) $X'=F(t,X)$ for large t satisfying, as $t\to\infty$, (2) $\lim X(t)=0$ and (3) $\lim X(t)/|X(t)|=Y_0$ exists. Let $F(t,X)$ be defined for large $t>0$ and small $|X|$ and let $F(t,0)\equiv0$. Suppose that Y, Y_0 are unit vectors, that $G(Y)=\lim r^{-1}F(t,rY)$, as $(r,t)\to(0,\infty)$, exists uniformly for Y near Y_0, that $G(Y_0)\cdot Y_0<0$, and that $G(Y)$ is not parallel to Y except for $Y=Y_0$. Let Z denote a vector satisfying $Z\cdot Y_0=1$ and consider X of the form $X=\rho Z$, $\rho\geq0$. Let C_0 be the cone $\{X:0<\rho\leq\rho_0,\ 0<|Z-Y_0|\leq\varepsilon_0\}$ and $D_0=\{Z:0<|Z-Y_0|\leq\varepsilon_0\}$. Suppose that there is a constant $a>0$ such that for $Z\in D_0$ either (i) $\Psi(Z)<-a(|Z|^2-1)$ or (ii) $\Psi(Z)>a(|Z|^2-1)$, where $\Psi(Z)=Z\cdot G(Z)-|Z|^2Y_0\cdot G(Z)$. In the case (i), all solutions entering C_0 for large t satisfy (2) and (3). In the case (ii), there is a 1-parameter family of solutions satisfying (2) and (3). This paper is related to results of Hartman and Wintner [Amer. J. Math. **75** (1953), 117–126; MR **14**, 645] for the two-dimensional conservative case.

P. Hartman (Baltimore, Md.)

7329:

Barrett, John H. Disconjugacy of second-order linear differential equations with nonnegative coefficients. Proc. Amer. Math. Soc. **10** (1959), 552–561.

The author considers differential equations of the form

(*) $\qquad (p(x)y')'+f(x)y=0 \quad (a\leq x<\infty),$

where $p(x)$ and $f(x)$ are positive and satisfy such regularity conditions as to guarantee absolutely continuous solutions of (*). Equation (*) is said to be disconjugate in (a,∞) if a solution $y(x)$ for which $y(a)=0$ does not vanish in (a,∞). If $p(x)\equiv1$— or, more generally, if $\int^\infty p^{-1}dx=\infty$ — disconjugacy in (a,∞) implies that the derivative of such a solution cannot vanish in (a,∞). If $\int^\infty p^{-1}dx<\infty$, this is not necessarily true, and there may exist in (a,∞) a focal point in the sense of the calculus of variations. The author extends some previously known disconjugacy criteria, which were derived for "disfocal" equations (*), to equations possessing focal points in (a,∞).

Z. Nehari (Pittsburgh, Pa.)

7330:

Aramă, Oleg. Le problème bilocal et le théorème des inégalités différentielles à nœuds confondus de S. A. Čaplygin pour des équations différentielles linéaires du 2e ordre. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 7–38. (Romanian. Russian and French summaries)

Soit Y la famille des intégrales dans (a,b) de l'équation $L(y)\equiv y''+p(x)y'+q(x)y=r(x)$ dont les coefficients sont continus dans (a,b). Par définition, Y a la propriété I_2 [resp. N_2] dans (a,b) si pour tous $x_1,x_2\in(a,b)$ et y_1,y_2 réels il existe une et une seule $y(x)\in Y$ telle que $y(x_i)=y_i$, $i=1,2$ [resp. pour toutes $y_1(x),y_2(x)\in Y$ distinctes, l'équation $y_1(x)=y_2(x)$ a au plus une solution dans (a,b)]. Le résultat principal de l'A. est que Y ait la propriété I_2 [ou N_2] sur (a,b) si et seulement si pour tout $x_0\in(a,b)$ et $u(x)\in C^2(a,b)$ les conditions $u(x_0)=u'(x_0)=0$, $L(u)>0$ (dans (a,b)) entraînent $u(x)>0$ pour tout $x_0\neq x\in(a,b)$. On donne une caractérisation des opérateurs L ayant cette propriété, de même que des limitations pour l'intervalle $(a,a+h)$ maximum dans lequel Y jouit de la propriété I_2. Le cas d'une équation d'ordre supérieur est aussi envisagé. *C. Foiaş* (Bucharest)

7331:

de Castro Brzezicki, A. Investigations in non-linear mechanics. Rev. Mat. Hisp.-Amer. (4) **18** (1958), 99–112, 146–156, 181–200. (Spanish)

This paper gives a brief and essentially standard exposition of the nature and purposes of the theory of nonlinear oscillations. It also gives some particular results which were new in 1954, when the paper was written. Thus the author considers various differential equations which are specializations of the general equation $\ddot{x}+f(x,\dot{x})\dot{x}+$

$g(x)h(\dot{x})=0$, and gives sets of conditions which are sufficient for the existence, and for the existence, uniqueness, and stability of periodic solutions. These sets of conditions, resembling those which are to be found in many places in the literature, involve too many details for convenient statement in a review. A procedure is described for obtaining numerical estimates of the amplitude of a periodic solution. Considering in particular the equation $\ddot{x}+f(x)\dot{x}+g(x)=0$, and assuming the existence of a periodic solution $x(t)$, the author multiplies the equation through by various functions of x and $\dot{x}$, integrates over a period, and so obtains a variety of results concerning mean values. {In a paper in Bell System Tech. J. **39** (1960), 365–367, the reviewer makes use of a similar procedure in connection with a system of partial differential equations.} Finally, the author considers the equations $\ddot{x}+f(x)\dot{x}+g(x)=e(t)$, $\ddot{x}+f(\dot{x})\dot{x}+g(x)=e(t)$, where $e(t)$ is periodic with the period T, and gives sets of conditions which are sufficient for the existence, uniqueness, and stability of periodic solutions with the period T. The proofs here depend upon the Brouwer fixed-point theorem.

L. A. MacColl (New York, N.Y.)

7332:

Pliss, V. A. The number of periodic solutions of equations whose right-hand member is a polynomial. Dokl. Akad. Nauk SSSR **127** (1959), 965–968. (Russian)

Some theorems are presented concerning the number of periodic solutions of equations of the form

$$(*) \qquad dy/dx = \sum_{k=0}^{n} p_k(x)y^{n-k},$$

where x is real, $p_0(x)=1$ and the other $p_k(x)$ are real, have the period ω and satisfy $|p_k(x_1)-p_k(x_2)| \leq L|x_1-x_2|$, $k=1, 2, \cdots, n$. If $y(x, c)$ denotes the solution of (*) such that $y(0, c)=c$ where c in general is complex, then y is periodic with period ω if and only if c is a zero of $f(c)=y(\omega, c)-c$. A periodic solution is counted as many times as the multiplicity of the corresponding c as a zero of f, and the author gives a number B related to the bound on the $|p_k|$ such that if $\omega \leq B$, then (*) has precisely n solutions of period ω. He also gives an example of (*) with $n=4$ which he says must have at least 6 distinct real periodic solutions.

C. E. Langenhop (Ames, Iowa)

7333:

Ogurtsov, A. I. The stability of solutions of two nonlinear differential equations of the third and fourth orders. J. Appl. Math. Mech. **23** (1959), 247–251 (179–181 Prikl. Mat. Meh.).

The "asymptotic stability for arbitrary initial perturbations" of the zero solution of

$$(1) \qquad \dddot{\xi}+f(\ddot{\xi})+b\dot{\xi}+a\xi = 0$$

is proved under the following assumptions: $f(0)=0$, f' exists for all real numbers, $f'>a/b$, where $a>0$ and $b>0$ are real numbers. This generalises somewhat a result obtained by V. A. Pliss [Dokl. Akad. Nauk SSSR **111** (1956), 1178–1180; MR **18**, 899]. For the proof the author replaces the differential equation by the equivalent system

$$\dot{x} = y, \quad \dot{y} = z-f'(x)y, \quad \dot{z} = -by-ax$$

and applies results of E. A. Barbašin and N. N. Krasovskiĭ [ibid. **86** (1952), 453–456; MR **14**, 646] that generalise the second method of Liapunov for arbitrary initial conditions. Using the same method, the same result is obtained for the zero solution of

$$(2) \qquad \xi^{(IV)}+f(\dddot{\xi})+c\ddot{\xi}+b\dot{\xi}+a\xi = 0$$

under the assumptions $f(0)=0$, $f'>0$ and exists for all real numbers, $bcf'-b^2-a(f')^2>0$, where a, b, c are real numbers with $a>0$ and $b>0$.

K. Matthies (Columbia, S.C.)

7334:

Ura, Taro. The problem of the extension of characteristic curves and stability. I. Sûgaku **9** (1957/58), 137–148. (Japanese)

This is a preliminary article for part II, reviewed below. This half is an introduction to some basic concepts and theorems like the existence theorem and uniqueness theorems for ordinary differential equations. Later some set theoretical discussions are made about the extensions and limiting point sets of trajectories of points belonging to a region under discussion.

H. Yamabe (Minneapolis, Minn.)

7335:

Ura, Taro. The problem of the extension of characteristic curves and stability. II. Sûgaku **9** (1957/58), 218–235. (Japanese)

This is a continuation of part I, above. The discussion is limited to the case of a two-dimensional region. At first the theory and classification of singular points as given by Poincaré and Bendixson is introduced. Later the successive extension of trajectories of points belonging to a given set is defined up to any ordinary number. Cantor's normal form on an ordinary number is essentially used.

Given an invariant region, the minimum of the ordinary numbers such that the extension of the orbit of the given invariant region differs from the original region, is proved to be a limiting number. When we write this number in the form of an exponent of ω, the exponent is called the order (or degree or rank, as the translation may vary) of the stability. Numerous examples are given to illustrate various situations.

In its content the paper seems, to the reviewer at least, to be a more abstract version of the author's previous paper, Ann. Sci. École Norm. Sup. (3) **70** (1953), 287–360 [MR **16**, 247].

H. Yamabe (Minneapolis, Minn.)

7336:

Leontovič, E. A. Certain analogies between plane algebraic curves and algebraic dynamical systems on a plane. Dokl. Akad. Nauk SSSR **129** (1959), 503–506. (Russian)

Gudkov [same Dokl. **98** (1954), 521–524; MR **16**, 741] has established two propositions in the theory of plane algebraic curves. In this paper two analogous propositions in the theory of plane algebraic dynamical systems are presented. The results are related to those of Duff [Ann. of Math. (2) **57** (1953), 15–31; MR **14**, 751] on the behavior of a limit cycle of a dynamical system as the vector field is rotated.

Specifically, let (D_n) denote the system,

$$\dot{x} = P(x, y) = \sum_{i+j+k=n} A_{ijk}x^i y^j,$$

$$\dot{y} = Q(x, y) = \sum_{i+j+k=n} B_{ijk}x^i y^j,$$

where n is fixed and the A_{ijk}, B_{ijk} are real and not all zero. Two such systems are said to be equivalent if the right hand sides differ only by a common non-zero factor. Then there is a 1-1 correspondence between the dynamical system (D_n) and the points of a metric projective space R_N $(N=n(n+3))$, which points are determined by the various sets of coefficients A_{ijk}, B_{ijk}. Let the system (D_n^μ) have the form of a rotated vector field: $\dot{x}=P_0(x, y)+\mu Q_0(x, y)$, $\dot{y}=Q_0(x, y)+\mu P_0(x, y)$, where the coefficients of $P_0(x, y)$ and $Q_0(x, y)$ are denoted by A^0_{ijk} and B^0_{ijk} respectively, $i+j+k=n$. In R_N the system (D_n^μ) defines the line d: $A_{ijk}=A^0_{ijk}+\mu B^0_{ijk}$, $B_{ijk}=B^0_{ijk}+\mu A^0_{ijk}$. Let λ be an arc without contact through a point p of L, a limit cycle of (D_n). Let v be a linear parameter such that p is given by $v=0$. Let $f(v)=\bar{v}$ be the function defined (for v small enough) by successive intersections of λ and trajectories of (D_n). If $F(v)=\bar{v}-v$, L is called a multiple limit cycle when $F(0)=0=F'(0)$; it is a double cycle if $F(0)=0=F'(0)$, but $F''(0)\neq 0$. A cycle is multiple if and only if its index $T^{-1}\int_L (P_x'+Q_y')dt$ vanishes.

Theorem 1: Let the system (D_n^0) have a double limit cycle L_0. Then: (1) each system (D_n) close enough to (D_n^0)—considering (D_n) and (D_n^0) as points p_n and p_n^0 in R_N—either has no multiple limit cycle in a suitable neighborhood of L_0, $U(L_0)$, or else has in $U(L_0)$ a unique double limit cycle; (2) the systems (D_n) close enough to (D_n^0) and having double limit cycles in $U(L_0)$ generate a plane $\psi(A_{ijk}, B_{ijk})=0$ in R_N passing through the point p_n^0 for which not all first derivatives vanish at p_n^0; the points of the line d corresponding to small positive μ lie in a region where $\psi(A_{ijk}, B_{ijk})>0$, while those corresponding to small negative μ lie in a region where $\psi(A_{ijk}, B_{ijk})<0$.

Two dynamical systems (D_{n1}) and (D_{n2}) are said to be ε-identical in a closed region $\bar{g}$ of the plane if there is a topological transformation of $\bar{g}$ onto itself mapping arcs of trajectories of (D_{n1}) onto arcs of trajectories of (D_{n2}), and conversely, such that the distance between two corresponding points is less than ε. Theorem 2: Let (D_n^0) have a double limit cycle L_0. Then: (1) for $\mu_1<0<\mu_2$ small enough, $(D_n^{\mu_1})$ and $(D_n^{\mu_2})$ are not ε-identical in any region $\bar{g}\subset U(L_0)$ containing L_0—in fact one system has no limit cycle in $\bar{g}$ while the other has two; (2) for each $\varepsilon>0$ any two systems $(D_n^{\mu_1})$ and $(D_n^{\mu_2})$ are ε-identical in some region $g_0\subset U(L_0)$ containing L_0 if μ_1 and μ_2 are small enough and $\mu_1\mu_2>0$.

The paper contains a number of notational errors.

C. S. Coleman (Claremont, Calif.)

7337:

Kukles, I. S. On Frommer's method in singular point investigations. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 367–370. (Russian)

The author considers the equation

$$dy/dx = [Y_n(x, y)+Y(x, y)]/[X_n(x, y)+X(x, y)],$$

where X_n, Y_n are homogeneous polynomials of degree n, whereas X and Y are analytic of higher order than n. The problem is to classify the behavior of the characteristics near the origin, and in particular to determine when, and how many, characteristics enter the origin with a definite tangent. Many detailed results are presented, which are stated in terms of the notions of order and measure of curvature of characteristics entering the origin. These notions were used by M. Frommer [Math. Ann. **99** (1928), 222–272] in an earlier study of this problem. Certain canonical equations are given which serve to distinguish the various cases. A rather complete treatment of this problem has been given by Lefschetz [Bol. Soc. Mat. Mexicana (2) **1** (1956), 13–27; ibid. (2) **2** (1957), 63–74; MR **18**, 481; **21** #2793] and Barocio [*Contributions to the theory of nonlinear oscillations*, vol. 3, pp. 127–135, Princeton Univ. Press, Princeton, N.J., 1956; MR **19**, 145].

E. A. Coddington (Los Angeles, Calif.)

7338:

Kukles, I. S. On characteristics reaching the origin with zero or infinite orders or measures of curvature. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 1, 15–27. (Russian. Uzbek summary)

Ce travail est une suite directe du travail de l'A. "Sur la méthode de Frommer" [mêmes Izv. **1957**, no. 4, 85–95]; on trouve tous les ordres et toutes les mesures de courbure des caractéristiques (qui passent par l'origine) de l'équation (*) $dy/dx=Y(x, y)/X(x, y)$, et on donne les conditions nécessaires et suffisantes pour que les ordres et mesures de courbure des caractéristiques des équations (*) et

$$du/dx = [f_1(x, u)-\lambda u f_2(x, u)]/x f_2(x, u)$$

soient zéro ou infinité.

A. Švec (Prague)

7339:

Kukles, I. S. Three discrimination problems. Dokl. Akad. Nauk SSSR **128** (1959), 239–242. (Russian)

The author considers the equation

$$\frac{dy}{dx} = \frac{Y_n(x, y)+Y(x, y)}{X_n(x, y)+X(x, y)},$$

where X_n, Y_n are homogeneous polynomials of degree n, and X, Y satisfy a Lipschitz condition in both variables in some neighborhood of the origin, and moreover $X(x, y)/r^n\to 0$, $Y(x, y)/r^n\to 0$, as $r=\sqrt{(x^2+y^2)}\to 0$. In polar coordinates this equation takes the form

$$r\frac{d\varphi}{dr} = \frac{F(\varphi)+f(r, \varphi)}{G(\varphi)+g(r, \varphi)},$$

where F, G are homogeneous polynomials of degree n in $\cos\varphi$ and $\sin\varphi$, and f, g are continuous functions which satisfy a Lipschitz condition in φ. Suppose $F(\varphi_0)=0$ and $G(\varphi_0)\neq 0$, and let $F(\varphi)/G(\varphi)=A(\varphi-\varphi_0)^k+a_1(\varphi-\varphi_0)^{k+1}+\cdots$. Three cases are distinguished: (1) $A<0$, k odd; (2) $A\neq 0$, k even; (3) $F(\varphi)\equiv 0$. In case (1) either there exists one characteristic, or infinitely many characteristics, which tend to the origin in the direction φ_0. Similar alternatives are valid in the other two cases. The three discrimination problems correspond to discriminating between the two alternatives in each of the three cases (1)–(3). Theorems are stated which give sufficient analytic conditions for such a discrimination in the three cases.

E. A. Coddington (Los Angeles, Calif.)

7340:

Vasil'eva, A. B. The asymptotic behaviour of solutions of some boundary problems for quasilinear equations involving a small parameter in the highest derivative term. Dokl. Akad. Nauk SSSR **123** (1958), 583–586. (Russian)

The author considers the quasi-linear equation

$$\mu y'' = -A(t, y)y' + B(t, y),$$

where μ is a non-negative parameter, together with two sets of boundary conditions: (i) $y'(0)=0$, $y(1)=0$; (ii) $y(0)=0$, $y(1)=0$. The degenerate equation is obtained by letting $\mu=0$, $A(t, u)u' = B(t, u)$. This is considered with the initial condition $u(1)=0$. The problem is to obtain estimates for the solutions of the non-degenerate boundary-value problems in terms of the solution of the degenerate problem. It is indicated how approximations of the order of μ^2 may be obtained for the conditions (i), and of the order μ for the conditions (ii). Higher order approximations may then be computed. {The author does not mention the earlier work of W. Wasow [Comm. Pure Appl. Math. **9** (1956), 93–113; MR **18**, 39], in which he develops the solution in a convergent and asymptotic series.} *E. A. Coddington* (Los Angeles, Calif.)

7341:

Vasil'eva, A. B. Uniform approximation to the solution of a set of simultaneous differential equations involving a small parameter in the derivative and its application to boundary value problems. Dokl. Akad. Nauk SSSR **124** (1959), 509–512. (Russian)

This is a sequel to the author's paper reviewed just above, and presents refinements of her procedure for obtaining approximations to solutions of boundary value problems of the type $\mu z' = F(z, y, t)$, $y' = f(z, y, t)$ $(\mu \geqq 0)$; $z(t_0) = z_0(\mu)$, $y(t_0) = y_0(\mu)$.

E. A. Coddington (Los Angeles, Calif.)

7342:

Hudaĭ-Verenov, M. G. On the number of limit cycles of the system

$$dy/dx = Q(x, y, z)/P(x, y, z), \quad dz/dx = R(x, y, z)/P(x, y, z),$$

where $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ are second degree polynomials. Dokl. Akad. Nauk SSSR **128** (1959), 899–902. (Russian)

Petrovskiĭ and Landis have given upper bounds for the number of limit cycles of the system $dy/dx = P(x,y)/Q(x, y)$, where $P(x, y)$ and $Q(x, y)$ are polynomials [Mat. Sb. (N.S.) **37** (**79**) (1955), 209–250; **43** (**85**) (1957), 149–168; MR **17**, 364; **19**, 746]. In this paper the author uses the ideas and techniques of Petrovskiĭ and Landis to discuss the problem given in the title ((1) will denote the system given there).

Considering (x, y, z) in a complex projective space, let R_6 denote the corresponding 6-dimensional real space with a compact metric. A solution of (1), $y = \varphi(x)$, $z = \psi(x)$, is a complete analytic function whose graph Φ in R_6 is called an integral curve. A 1-dimensional closed curve on a Φ is called a cycle (a limit cycle of (1) is a cycle on some Φ). In the space of the coefficients of P, Q, and R, let α_0 correspond to the system (2) $dy/dx = -y(y-1)x^{-1}(x-1)^{-1}$, $dz/dx = -z(z-1)x^{-1}(x-1)^{-1}$. The general solution of (2) is $y = c_1(x-1)(x-c_1)^{-1}$, $z = c_2(x-1)(x-c_2)^{-1}$. Let $N(m)$ be the number of closed curves with at most m self-intersections in the x-plane (with 0, 1, c_1, c_2 deleted) which are not homotopic to one another. For a system α, r cycles $L_1, \cdots, L_r$ lying on integral curves of α are said to have property P_m if (1) L_i not ~ 0, L_i not $\sim L_j$ $(i, j = 1, \cdots, r, i \neq j)$, and (2) the projection of each L_i on the x-plane has no more than m self-intersections.

The author gives two results: 1° The number of cycles with property P_m for a system α lying in a small neighborhood of α_0 (with the exception of a manifold of lower dimension) is no more than $N(m)\cdot 16 - (m+1)\cdot 33$; 2° the number of cycles with property P_m for a system α in the neighborhood given in 1° above is a constant (= the number of solutions of a certain set of algebraic equations). *C. S. Coleman* (Claremont, Calif.)

7343:

Zverkin, A. M. On the theory of linear differential equations with a lagging argument and periodic coefficients. Dokl. Akad. Nauk SSSR **128** (1959), 882–885. (Russian)

Let (*) denote the equation

$$X'(t) + A_0(t) + A_1(t)X(t-\tau_1) + \cdots + A_m(t)X(t-\tau_m) = 0,$$

where $X(t)$ is an n-vector, each $n \times n$ matrix $A_i(t)$ has period T, and each $\tau_i > 0$. Three questions originally posed for an equation without lagging argument are considered in this paper.

(1) Is there a linear change of variables reducing (*) to an equation with constant coefficient matrices? The author answers this in the negative with a counter example, but he gives particular types of equations for which such a reduction is possible.

(2) For $n=1$ [n arbitrary] are there functions $f(t)$ [$V(t)$] of period T for which $\exp(\int_0^t f(\xi)d\xi)$ [$V(t)e^{\lambda t}$] are solutions of (*)? If so, how are these functions determined? Is the set of such solutions a fundamental set for (*)? The author shows how to find these functions, in the process introducing sets of eigenvalues for (*). However, again by a counter example, he shows that these functions do not generate a set of fundamental solutions of (*) in general.

(3) Do the eigenvalues obtained in (2) correspond to the Liapunov numbers of the solutions of (*)? The author asserts that the Liapunov numbers are precisely the eigenvalues with the signs of the real parts reversed—this in spite of the negative answer to the last question posed in (2). *C. S. Coleman* (Claremont, Calif.)

7344:

Levin, J. J. On the matrix Riccati equation. Proc. Amer. Math. Soc. **10** (1959), 519–524.

The author considers the matrix Riccati differential equation $\Gamma' + \Gamma G_3 \Gamma + \Gamma G_4 - G_1 \Gamma - G_2 = 0$, where G_1, G_2, G_3, and G_4 are $n_1 \times n_1$, $n_1 \times n_2$, $n_2 \times n_1$, and $n_2 \times n_2$ matrices, respectively, and Γ is $n_1 \times n_2$. He relates Γ to solutions of the matrix-vector differential equation $\psi' = G\psi$, where $G = \begin{pmatrix} G_1 G_2 \\ G_3 G_4 \end{pmatrix}$. For $n_1 = n_2$ he obtains a generalization of the classical cross-ratio property of solutions of the scalar Riccati equation.

W. J. Coles (Salt Lake City, Utah)

7345:

Mlak, W. Limitations and dependence on parameter of solutions of non-stationary differential operator equations. Ann. Polon. Math. **6** (1959), 305–322.

The problems discussed are the uniqueness, the stability and the existence of solutions of the differential equation $dx/dt = A(t)x + f(t, x)$ in a Banach space; $A(t)$ is a closed linear operator with domain $D(A(t))$ and range in E, and d/dt the strong derivative with respect to the time parameter t. The author introduces [same Ann. **3** (1956), 37–40; MR **18**, 737], with T. Ważewski [Ann. Soc. Pol. Math. **23** (1950), 112–166; MR **12**, 705], the epidermic condition (T) for a non-negative function $\sigma(t, u)$ which is continuous for $t \in [0, \alpha]$ and $u \geq 0$: The right maximal solution $\omega(t, u_0)$ of the differential equation $u' = \sigma(t, u)$, $\omega(0, u_0) = u_0$, exists in $[0, \alpha]$ for every $u_0 \geq 0$. The uniqueness theorem 6 reads: For every $t \in [0, \alpha]$, let $D(A(t))$ be dense in E and $\|\lambda(\lambda I - A(t))^{-1}\| \leq 1$ for $\lambda \geq \lambda(t)$. Let $\|f(t, x) - f(t, y)\| \leq \sigma(t, \|x - y\|)$ with a σ satisfying (T). Then, through each point $(0, x_0)$, $x_0 \in E$, there passes at most one solution of the equation $dx/dt = A(t)x + f(t, x)$, $t \in [0, \alpha)$. The stability theorem 4 reads: Let $A(t)$ and σ satisfy the same conditions as in theorem 6, and $\|f(t, x)\| \leq \sigma(t, \|x\|)$. Then the continuous solution of $d^+x/dt = A(t)x + f(t, x)$ satisfies the estimate $\|x(t)\| \leq \omega(t, \|x(0)\|)$. Theorem 15 is concerned with derivability with respect to the parameter λ of the equation $dx/dt = A(t)x + f(t, x, \lambda)$. Theorem 13 gives an existence proof of the solution of the equation $dx/dt = A(t)x + f(t, x)$ when E is a Hilbert space and A is self-adjoint with $(Ax, x) \leq -(x, x)$.

K. Yosida (Tokyo)

7346:

Hukuhara, Masuo. Théorèmes fondamentaux de la théorie des équations différentielles ordinaires dans l'espace vectoriel topologique. J. Fac. Sci. Univ. Tokyo. Sect. I 8 (1959), 111–138.

Let $\mathfrak{R}$ be a complete locally convex topological linear space over the real field R. Let f be a function on $R \times \mathfrak{R}$ to $\mathfrak{R}$. The author's main aim is to study differential equations of the form $dx/dt = f(t, x)$ and to extend to this realm of generality the classical results concerning majorant and minorant solutions of such equations. A number of results are obtained which cannot be described here. Among other things, solutions of the above differential equation are compared with solutions of the equation $dX/dt = F(t, X)$, where F is a real-valued function on $R \times R$ and conditions such as

$$|f(t, x)|_V \leq F(t, |x|_V)$$

are satisfied. (Here $|x|_V$ denotes the pseudo-norm generated by a convex symmetric neighborhood of the origin.) A number of uniqueness theorems are obtained.

R. G. Bartle (Urbana, Ill.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 7288, 7369, 7445, 7521, 7674, 7675, 7676.

7347:

Berutti Onesti, Natalia. Un teorema di esistenza per l'equazione $F(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$. Ann. Scuola Norm. Sup. Pisa (3) **13** (1957), 89–114.

The Cauchy problem for the equation of the title is solved under relatively weak conditions, which are too detailed to be mentioned here. An estimate for the domain of existence of the solution so constructed is also given.

C. R. DePrima (Pasadena, Calif.)

7348:

Protter, M. H. Vibration of a nonhomogeneous membrane. Pacific J. Math. **9** (1959), 1249–1255.

Let D be a simply-connected domain in the xy-plane, C its boundary, and λ_0 the lowest eigenvalue of the differential equation $u_{xx} + u_{yy} + \lambda p(x, y) = 0$ $(p(x, y) \geq 0)$ with the boundary condition $u = 0$ on C. If $P(x, y)$, $Q(x, y)$ are C^1 in D, and if $a(x, y)$ is C^2 and positive, the author shows that $\lambda_0 \geq \mu$ if μ is a number such that $P_x + Q_y \geq a^{-1}(P^2 + Q^2) + 2^{-1}(a_{xx} + a_{yy}) + ap\mu$ throughout D. By suitable choices of the functions P, Q, a, a number of useful lower bounds for λ_0 are obtained.

Z. Nehari (Pittsburgh, Pa.)

7349:

Kato, Tosio. Growth properties of solutions of the reduced wave equation with a variable coefficient. Comm. Pure Appl. Math. **12** (1959), 403–425.

Let D_L be the set of complex-valued functions u over x in real n-space R_n with $|x| = [\sum_{j=1}^n (x_j)^2]^{1/2} \geq L$, fixed $L < +\infty$, and such that u possesses continuous partials of all orders ≤ 2 there. For $u \in D_L$ define

$$[Hu](x) = -\nabla^2 u(x) + q(x)u(x)$$

over $|x| \geq L$; here ∇^2 is the Laplacian and $q(x)$ is a fixed measurable, real-valued function over $|x| \geq L$ for which the finite $\lim_{|x| \to \infty} q(x) = c$ exists, with c taken to be 0 for convenience. Defining

$$\eta_\lambda = (2\sqrt{\lambda})^{-1} \lim_{r \to +\infty} \left(\sup_{|x| \geq r} |x| q(x) \right)$$

for constant real $\lambda > c = 0$, in the first part of this paper the author proves for such $q(x)$ continuous over $|x| \geq L$ the important result that either (i) $\eta_\lambda < \frac{1}{2}$ alone, or both (ii) $\eta_\lambda < 1$ and (iii) $\int_L^\infty \{\sup_{|x| = r} |q(x)|\} dr < +\infty$, are sufficient to prove the nonexistence of $u \in D_L$ such that $[Hu](x) = \lambda u(x)$ over $|x| \geq L$ and that $0 < \int_{|x| \geq L} |u(x)|^2 d\mu_n(x) < +\infty$, μ_n denoting n-dimensional Lebesgue measure. Earlier [Ann. of Math. **54** (1951), 554–594; MR **13**, 847; Th. 6.7, p. 586 and Th. 6.8, p. 591] the reviewer proved by using the classical Weyl argument that for purely radial $q(x)$ (i.e., $q(x) = q(|x|)$) condition (iii) alone suffices for the above conclusion, but for nonradial $q(x)$ he could obtain the conclusion only under exponential bounds at ∞ much more stringent than (i). For $n = 3$, $L = 0$, Miranker [Comm. Pure Appl. Math. **10** (1957), 491–502; MR **20** #637] also obtains the above conclusion from the single condition

$$\text{(iv)} \qquad \int_{R_3} (4\pi|x|)^{-1} |q(x)| d\mu_3(x) < 1.$$

Actually under the slightly stronger single condition

$$\text{(iv')} \qquad \left[\operatorname*{ess\,sup}_{y \in R_n} \int_{R_n} (4\pi|x - y|)^{-1} |q(x)| d\mu_3(x) \right] < 1$$

the reviewer earlier noted [Bull. Amer. Math. Soc. **60** (1954), 346; abstract 430] that his 1951 construction [see above] easily shows that his constructed self-adjoint H'

in $L_2(R_3)$ associated with H has its spectrum purely continuous and exactly the set $[0, +\infty)$, which implies the above conclusion but also unfortunately suppresses the possible point spectrum in $(-\infty, 0)$. Wienholtz [Math. Ann. **135** (1958), 50–80; MR **20** #1090; particularly pp. 77–80] also obtains the above conclusion under different conditions which undesirably include differentiability of $q(x)$ to order 3 or more.

The remaining part of this paper is quite independent of the first part. It obtains order estimates at ∞ of $u \in D_L$ satisfying $Hu = \lambda u$, in case $q(x)$ (now allowed to be complex valued) and complex λ have the set of values of $\lambda - q(x)$ essentially lying to the left, in the complex plane, of some straight line crossing the real axis at the origin.

F. H. Brownell (Seattle, Wash.)

7350:

Titchmarsh, E. C. On the eigenvalues in problems with spherical symmetry. III. Proc. Roy. Soc. London. Ser. A **252** (1959), 436–444.

[For part II, see same Proc. **251** (1959), 46–54; MR **21** #2807.] "Let $N(\lambda)$ denote the number of eigenvalues not exceeding λ of the three-dimensional equation $\nabla^2\psi + \{\lambda - q(r)\}\psi = 0$ over the whole space. The problem of the behaviour of $N(\lambda)$ as $\lambda \to \infty$ is considered in the case where $q(r) = r^c$, c being a constant. It is shown that if $c = 4$ or 6, $N(\lambda) = a\mu^3 + b\mu^2 + O(\mu^{5/2})$, where $\mu = \lambda^{(c+2)/(2c)}$ and a and b are constants. This result is derived from a theorem due to van der Corput in a region of general type. It does not hold in the case $c = 2$ which is exceptional." (Author's summary)

E. T. Copson (St. Andrew)

7351:

Flitman, L. M. On a mixed boundary value problem for the wave equation. J. Appl. Math. Mech. **22** (1958), 1185–1191 (829–832 Prikl. Mat. Meh.).

The function $u(x, y, t)$ defined in the half-plane $y > 0$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$$

with the boundary conditions

$$u(x, t) = 0 \quad (|x| > 1,\ y = 0),$$

$$\frac{\partial u}{\partial y}(x, t) = a(x, t) \quad (|x| < 1,\ y = 0).$$

This is the plane problem of acoustic radiation from a strip. On $y = 0$ the values of $u(x, t)$ in $|x| < 1$, and of $\partial u/\partial y$ in $|x| > 1$ are to be found. The solution proceeds by iteration from the analogous problem for a semi-infinite strip (which is solved in the paper). The problem with the boundary conditions

$$u(x, t) = a(x, t) \quad (|x| < 1,\ y = 0),$$

$$\frac{\partial u}{\partial y}(x, t) = 0 \quad (|x| > 1,\ y = 0).$$

is also solved. [Additional reference: F. G. Friedlander, *Sound pulses*, Cambridge University Press, New York, 1958; MR **20** #3703; cf. § 5.7.]

F. Ursell (Cambridge, England)

7352:

Maz'ya, V. G. Solution of Dirichlet's problem for an equation of elliptic type. Dokl. Akad. Nauk SSSR **129** (1959), 257–260. (Russian)

This abstract contains various theorems relating to a-priori estimates, uniqueness and existence of generalized solutions of linear equations of the form

$$(1)\quad \mathfrak{M}u \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x)$$

and of non-linear equations of the forms

$$(2)\quad \mathfrak{N}u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = \varphi(x; u, u_{x_\nu}),$$

$$-\Delta u = \varphi(x; u, u_{x_\nu}; u_{x_\eta x_\mu}) \quad (\nu, \mu, \eta = 1, \cdots, n),$$

which are defined for x in a bounded closed region Ω, and for all values of u, u_{x_ν}, $u_{x_\eta x_\mu}$. It is assumed that

$$\sum_{i,j=1}^{n} a_{ij}\alpha_i\alpha_j \geq \mu^2 \sum_{i=1}^{n} \alpha_i^2, \quad \mu^2 = \text{const} > 0.$$

Solutions in the class $u(x) \in \mathring{W}_p^{(1)}(\Omega) \cap W_p^{(2)}(\Omega)$ $(1 \leq p \leq \infty)$ are considered. Let λ^2 be the first eigenvalue of the operator $\mathfrak{N}u$. If λ^2 is sufficiently large, depending in a simple way only on $c(x)$ and $\sum_{i=1}^{n} \partial b_i/\partial x_i$, an a-priori estimate of the form $\|u\|_{W_p^{(2)}(\Omega)} \leq c\|f\|_{L_p(\Omega)}$ is obtained for any solution $u(x)$ of (1) for suitable p, $1 < p < \infty$.

Consider next a solution of (2), and suppose that whenever $u > v$,

$$\varphi(x; u, u_\nu) - \varphi(x; v, v_\nu) \leq M_0(u - v) + \sum_{\nu=1}^{n} M_\nu|u_\nu - v_\nu|, \quad M_\nu = \text{const}.$$

Then if either $M_0 \leq 0$, or $M_0 \geq 0$ and $\lambda > M_0^{1/2} + (2\mu)^{-1}(\sum_{\nu=1}^{n} M_\nu^2)^{1/2}$, any solution of the Dirichlet problem will be unique in $W_{p,0}^{(2)}$. The coefficient of $M_0^{1/2}$ is sharp, and the coefficient $\frac{1}{2}$ in the second term cannot be replaced by anything smaller than $1/4\pi$. If, in addition,

$$|\varphi(x; u, u_\nu) - \varphi(x; v, v_\nu)| \leq M_0|u - v| + \sum_{\nu=1}^{n} M_\nu|u_\nu - v_\nu|,$$

$$\frac{M_0}{\lambda^2} + \frac{1}{\lambda}\left(\sum_{\nu=1}^{n} M_\nu^2\right)^{1/2} < 1,$$

then a solution exists in $W_{2,0}^{(2)}$.

Finally, consider equation (3) in a convex domain Ω, and suppose

$$|\varphi(x; u, u_\nu; u_{\eta\mu}) - \varphi(x; v, v_\nu; v_{\eta\mu})| \leq M_0|u - v| + \sum_{\nu=1}^{n} M_\nu|u_\nu - v_\nu| + \sum_{\eta,\mu=1}^{n} M_{\eta\mu}|u_{\eta\mu} - v_{\eta\mu}|,$$

$$\frac{M_0}{\lambda^2} + \frac{1}{\lambda}\left(\sum_{\nu=1}^{n} M_\nu^2\right)^{1/2} + \left(\sum_{\eta,\mu=1}^{n} M_{\eta\mu}^2\right)^{1/2} < 1.$$

Then there is a unique solution of the Dirichlet problem for (3) in Ω.

No proofs are given.

R. Finn (Stanford, Calif.)

7353:

Il'in, V. A. Solvability of a mixed problem for hyperbolic and parabolic equations on an arbitrary normal cylinder. Dokl. Akad. Nauk SSSR **127** (1959), 23–26. (Russian)

The author considers two problems:

$$(1)\qquad \begin{aligned} &L(u)-u_{tt} = -f(x,t) \quad \text{in} \quad \Omega_l = g\times[0 \leqq t \leqq l],\\ &u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad u|_\Gamma = 0; \end{aligned}$$

$$(2)\qquad \begin{aligned} &L(u)-u_t = -f(x,t) \quad \text{in} \quad \Omega_l,\\ &u(x,0) = \varphi(x), \quad u|_\Gamma = 0; \end{aligned}$$

where g is a n-dimensional domain with boundary Γ, L is a selfadjoint elliptic operator. Classical solutions (in a precisely defined sense) of problems (1), (2) are investigated.

O. A. Ladyženskaya has proved existence and uniqueness of these solutions under some sufficiently general assumptions about the operator L and functions $f(x,t)$, $\varphi(x)$, $\psi(x)$ and under rather special assumptions about the domain g. The main purpose of the author's work is to prove existence and uniqueness of these solutions for any normal cylinder Ω_l. (Ω_l is called normal if the domain g is normal; g is normal if Dirichlet's problem is solvable on g for any continuous function prescribed on the boundary of g.)

Only results without proof are presented in the paper; the whole work will be published later.

K. Rektorys (Prague)

7354:

Rothe, E. H. Some remarks on fundamental solutions of parabolic differential equations of second order. Michigan Math. J. **6** (1959), 227–245.

The author considers a parabolic equation (i) $\partial u/\partial t - L(u)=0$, where L is an elliptic selfadjoint operator in a n-dimensional simply connected domain D with the boundary $\dot{D}$. He defines a fundamental solution of (1) as a function $G(x,\xi,t)$ which (a) satisfies as function of x, t the equation (1); provided (b) for any function $h(x)$ continuous in the closure $\bar{D}$ of D, and for any subdomain D_1 of D,

$$(2)\qquad \lim_{t\to 0}\int_{D_1} h(\xi)G(x,\xi,t)d\xi = h(x) \quad \text{for} \quad x\in D_1,$$
$$= 0 \quad \text{for} \quad x\in \operatorname{Int}(D-D_1).$$

If D is bounded and $\dot{D}$ is smooth enough, then it is known that a fundamental solution of (1) may be written in the form

$$(3)\qquad G(x,\xi,t) = \sum_{k=1}^{\infty} u_k(x)u_k(\xi)e^{-\lambda_k t},$$

where $u_1(x), u_2(x), \cdots$ is a full orthonormal set of eigenfunctions of the problem

$$(4)\qquad L(u)+\lambda u = 0, \quad x\in D,$$

$$(5)\qquad u = 0, \quad x\in \dot{D},$$

and $\lambda_1, \lambda_2, \cdots$ are the corresponding eigenvalues.

The author refers then to two works of Dressel, who constructed the fundamental solution $\Gamma(x,\xi,t)$ of (1) in a totally different way. In the paper, the author shows that both fundamental solutions $G(x,\xi,t)$ and $\Gamma(x,\xi,t)$ have the same kind of singularities, i.e., that (6) $\gamma(x,\xi,t)= G(x,\xi,t)-\Gamma(x,\xi,t)$ is a regular function. He then presents some consequences of (6), concerning (a) asymptotic relations for eigenfunctions and eigenvalues of the problem (4), (5), (b) bilinear series of the form

$$\sum_{k=1}^{\infty} u_k(x)u_k(\xi)/\lambda_k,$$

(c) asymptotic behaviour of Γ and G for $t\to\infty$, and (d) limit relations between Green's functions belonging to a sequence of domains D_n.

K. Rektorys (Prague)

7355:

Nickel, Karl. Fehlerabschätzungen bei parabolischen Differentialgleichungen. Math. Z. **71** (1959), 268–282.

Consider the mixed boundary value problem for a nonlinear parabolic equation of the second order (E), the boundary conditions (C) being linear, and consider a second problem of the same nature, (E′), (C′). The author proves a general theorem which qualitatively states that if (E′) and (C′) approximate (E) and (C), respectively, then the solution of (E′), (C′) approximates the solution of (E), (C). The precise statement is too long to be given here. The author also applies his theorem to derive approximation-estimates in some specific problems.

A. Friedman (Minneapolis, Minn.)

7356:

Danilyuk, I. I. On the oblique derivative problem for the general quasilinear elliptical system of the first order. Dokl. Akad. Nauk SSSR **127** (1959), 953–956. (Russian)

Es handelt sich um eine Untersuchung von Richtungsableitungen von allgemeinen quasilinearen elliptischen Differentialsystemen erster Ordnung für zwei unbekannte Funktionen u, v über der x, y-Ebene, für welche der Verfasser in komplexer Schreibweise

$$(1)\qquad \frac{\partial w}{\partial \bar{z}}+\mu_1(z,w)\frac{\partial w}{\partial z}+\mu_2(z,w)\frac{\partial \bar{w}}{\partial z}+\nu(z,w) = 0 \qquad (w = u+iv),$$

$$\left(\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right)$$

ansetzt. Für die Koeffizienten wird für alle z aus dem Gebiet $\bar{D}$ und eine vorgegebene Schranke $|w|\leqq R$ die Gültigkeit der Ungleichung $|\mu_1|+|\mu_2|\leqq\mu_0+1$ vorausgesetzt. Dabei ist $\mu_0(R)$ eine gewisse Konstante. Auf dem Rand Γ des Gebietes D wird ein beliebiger Differentialoperator

$$(2)\qquad \tilde{\omega}(z,u,v,u_x,u_y,v_x,v_y) \equiv \operatorname{Re}\,\omega(z,w,w_z,w_{\bar{z}})$$

eingeführt, wobei $\tilde{\omega}$ und ω gegebene Funktionen ihrer Argumente bezeichnen. Jetzt wird eine komplexe Funktion w aus der Klasse $w_p{}^1(D)$, $p>2$ gesucht, welche im Innern von D der Gleichung (1) und am Rand Γ der Randbedingung (Richtungsbedingung) genügt. Der Weg zur Konstruktion einer solchen Funktion führt über die Behandlung eines Riemann-Hilbertschen Problems, die der Verfasser bereits bei früherer Gelegenheit gegeben hatte [cf. I. I. Danilyuk, dieselben Dokl. **122** (1958), 9–12, 175–178; MR **21** #1449; 1450]. Dabei wird an Stelle des Ausgangssystems das System

$$(3)\qquad \begin{aligned} &\frac{\partial F_1}{\partial \bar{z}}+\mu_1(z,F_1)F_2+\mu_2(z,F_1)\bar{F}_2+\nu(z,F_1) = 0,\\ &\frac{\partial F_2}{\partial \bar{z}}-F_2 = 0, \end{aligned}$$

$$\frac{\partial F_2}{\partial \bar{z}}+q_1(z,F_1)\frac{\partial F_2}{\partial z}+q_2(z,F_1)\frac{\partial \bar{F}_2}{\partial \bar{z}}+\nu_2(z,F_1,F_2) = 0,$$

mit den Randbedingungen

$$\text{(4)} \qquad \operatorname{Re}\omega(z, w_2, w_3) = 0, \quad \operatorname{Re}[F_1 - F_3] = 0, \quad \operatorname{Re}[iF_1 + iF_3] = 0$$

betrachtet und wiederum nach einer allgemeinen Lösung der Klasse $w_p{}^1(D)$ gefragt, welche dem System und den Randbedingungen genügt. Da die Lösung dieses Problems bekannt ist, kann man die ursprüngliche Aufgabe darauf zurückführen, indem man (3) in der komplexen Gestalt

$$\frac{\partial\varphi}{\partial\bar{z}} + Q_1(z,\varphi)\frac{\partial\varphi}{\partial z} + Q_2(z,\varphi)\frac{\partial\bar{\varphi}}{\partial\bar{z}} + \tau(z,\varphi) = 0$$

schreibt. Dabei ist φ ein dreikomponentiger komplexwertiger Vektor, ebenso τ. Die Koeffizienten Q_1 und Q_2 sind durch die Matrizen

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

gegeben und die Randbedingung gewinnt die Gestalt

$$\operatorname{Re}[g(z)\varphi(z)] = 0, \quad g = \begin{pmatrix} 0 & z^\varkappa & 0 \\ 1 & 0 & -1 \\ i & 0 & i \end{pmatrix}, \quad z \in \Gamma.$$

Für die Lösung des ursprünglichen Problems werden jetzt noch die Fälle $\varkappa \geqq 0$ und $\varkappa_1 = -\varkappa < 0$ $(\varkappa = |\varkappa_1|)$ unterschieden. *M. Pinl* (Cologne)

7357:

Mitchell, Josephine. Representation theorems for solutions of linear partial differential equations in three variables. Arch. Rational Mech. Anal. **3**, 439–459 (1959).

The close and mutually-profitable relation between classical complex variables and the study of Laplace's equation has, of course, inspired several attempts to establish an analogous relationship between function theory and the solutions of more general partial differential equations. A particularly sustained effort in this direction has been undertaken by Stefan Bergman and his collaborators [Bergman, Trans. Amer. Math. Soc. **73** (1952), 1–34; MR **14**, 382; Kreyszig, J. Rational Mech. Anal. **5** (1956), 805–820; MR **18**, 130]. The Bergman-Kreyszig-Mitchell approach depends upon replacing the operator "real part of" (which would map an analytic function into a harmonic function) with a Bergman integral operator, an example of which is given below.

In the present paper, an analytic function $f(u, \zeta)$ is mapped into a harmonic vector (i.e., an ordered triple of harmonic functions of the three variables x, y, and z) by the "three-dimensional" operator

$$H_1(X) = (2\pi i)^{-1}\int f(u,\zeta)(d\zeta)/\zeta,$$

$$H_2(X) = (2\pi i)^{-1}\int f(u,\zeta)[i(\zeta+\zeta^{-1})/2](d\zeta)/\zeta,$$

$$H_3(X) = (2\pi i)^{-1}\int f(u,\zeta)[(\zeta-\zeta^{-1})/2](d\zeta)/\zeta.$$

Here $u = u(X, \zeta) \equiv x + Z\zeta + Z^*\zeta^{-1}$, $Z = \frac{1}{2}(iy + z)$, $Z^* = \frac{1}{2}(iy - z)$, and $X = (x, y, z)$. The harmonic vector $\mathbf{H} \equiv (H_1, H_2, H_3)$ satisfies $\operatorname{div}\mathbf{H} = 0$, $\operatorname{curl}\mathbf{H} = 0$. Using this "integral operator" mapping, the author studies the in-the-large behavior of solutions, for the case where $f(u, \zeta)$ is an algebraic function

$$f(u, \zeta) \equiv p(u, \zeta, S)/q(u, \zeta),$$

p and q being polynomials, and u, ζ, and S being connected by an algebraic equation. *R. B. Davis* (Syracuse, N.Y.)

7358:

Malferrari, Angelo. Su certe equazioni quasi lineari di tipo parabolico di ordine superiore al secondo. Atti Sem. Mat. Fis. Univ. Modena **8** (1958/59), 174–216.

The author addresses himself to the fourth order nonlinear parabolic problem [cf. B. Pini, Rend. Sem. Mat. Padova **27** (1957), 319–349, 387–410; MR **20** #7150]:

$$L_0 u = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{xxx})$$

in $R: [0 < x < 1, 0 < y \leq h]$ with u and its first normal derivations assigned as continuous functions on the boundary of R with the obvious exception of the side $y = h$. Here

$$L_0 u = u_{xxxx} - 2a(x, y)u_{xxy} + u_{yy},$$

with $a(x, y) \geqq a_0 > 0$ in R and satisfying appropriate Hölder continuity properties in R.

Since it is possible that, for continuous boundary data, solutions of $L_0 u = f$ may have higher (than the first) derivatives which become infinite in a well-determined manner as the boundary is approached, the author sets up a class S of functions to take care of this eventuality. The a-priori bounds of Pini for $L_0 u = f$ are then sharpened accordingly, so that, with appropriate conditions on F, an existence (and uniqueness) theorem for sufficiently small h is obtained via a simple application of Rothe's version of the Schauder fix-point theorem.

C. R. DePrima (Pasadena, Calif.)

7359:

Pini, Bruno. Sulle equazioni lineari del quarto ordine in due variabili con caratteristiche coincidenti. I. Atti Sem. Mat. Fis. Univ. Modena **8** (1958/59), 130–166.

The author continues his study of boundary-value problems for the fourth order "parabolic" equation in two independent variables:

$$\text{(1)} \quad \alpha\frac{\partial^4 u}{\partial x^4} + 2\beta\frac{\partial^3 u}{\partial x^2 \partial y} + \gamma\frac{\partial^2 u}{\partial y^2} + \sum_{i=0}^{3}\alpha_i\frac{\partial^i u}{\partial x^i} + \delta\frac{\partial^2 u}{\partial x \partial y} + \varepsilon\frac{\partial u}{\partial y} = f,$$

α, β, γ, α_i, δ, ε, f are functions of x, y. In two previous papers [Rend. Sem. Mat. Univ. Padova **27** (1957), 319–349, 387–410; MR **20** #7150], the cases in which the roots λ_i of the quadratic equation $\alpha\lambda^2 + 2\beta\lambda + \gamma = 0$ satisfied $\operatorname{Re}(\lambda_i) > 0$ (or < 0) in the domain under consideration were treated in some detail. The present paper deals with the case in which the λ_i are real and $\lambda_1\lambda_2 < 0$. In this situation (1) has a normal form given by

$$\text{(2)} \qquad Lu = L_0 u + \sum_{i=0}^{2} a_i\frac{\partial^i u}{\partial x^i} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial u}{\partial y},$$

with

$$\text{(3)} \qquad L_0 u = \frac{\partial^4 u}{\partial x^4} + 2\alpha\frac{d^2 u}{\partial x^2 \partial y} - \frac{\partial^2 u}{\partial y^2}.$$

The problems considered are $Lu = f$ in $R\,[0 < x < 1, 0 < y < h]$ with u prescribed on ∂R and either $\partial^2 u/\partial x^2$ or $\partial u/\partial x$, respectively, prescribed on the vertical sides of R. The

author treats the first of these two problems in detail (with a = constant) by constructing the Green's function for the corresponding homogeneous problem associated with L_0 and then employing the method of integral equations to obtain solutions for the general boundary value problem associated with L under suitable assumptions on the coefficients of L and the given data. Since a uniqueness theorem need not hold for the general problem (it does for L_0), the desired existence theorem is in terms of a Fredholm alternative. The solution of the second of the two boundary value problems is sketched.

C. R. DePrima (Pasadena, Calif.)

7360a:

Pini, Bruno. Contributi allo studio dell'equazione delle vibrazioni della sbarra elastica. Atti Sem. Mat. Fis. Univ. Modena 8 (1958/59), 90–120.

7360b:

Pini, Bruno. Ulteriori contributi allo studio dell'equazione delle vibrazioni della sbarra elastica. Atti Sem. Mat. Fis. Univ. Modena 8 (1958/59), 217–227.

Another version of (1) [cf. previous review] comes under the author's scrutiny—namely, the case in which $\mathrm{Re}(\lambda_i) \equiv 0$, $i = 1, 2$. The normal form (2) there has $L_0 u = \partial^4 u/\partial x^4 + \partial^2 u/\partial y^2$. With y interpreted as a time coordinate (3) is then precisely the equation for the forced vibrations of an elastic bar. With this interpretation in mind, the author sets $b = 0$ in (2) and assumes all other coefficients are functions of x alone. Various initial-boundary value problems of bar theory are then solved (no conditions are prescribed on $y = h$ in this case) by the same methods employed by the author in his previous work on equation (1).

The second paper gives some uniqueness theorems for the problems discussed in the first paper.

C. R. DePrima (Pasadena, Calif.)

7361:

Pogorzelski, W. Propriétés des intégrales généralisées de Poisson-Weierstrass et problème de Cauchy pour un système parabolique. Ann. Sci. École Norm. Sup. (3) **76** (1959), 125–149.

In the first part of the paper the author deals with a parabolic system of N linear partial differential equations

$$(1) \quad \hat{\Psi}^{(\alpha)}(u_1, \cdots, u_N) \equiv \Psi^{(\alpha)}(u_1, \cdots, u_N) - \partial u_\alpha/\partial t = 0.$$

Here $\Psi^{(\alpha)}$ is a linear homogeneous differential operator of even order M, containing derivatives only with respect to the spatial variables $x_1, \cdots, x_n$, whose coefficients are real functions of $X = (x_1, \cdots, x_n)$ and t which are bounded for all X in the n-dimensional space E and for $0 \leq t \leq T$. The coefficients of the derivatives of highest order are assumed to satisfy Hölder conditions in both X and t, while the lower order coefficients satisfy Hölder conditions in X and are uniformly continuous in t. Making use of his matrix of fundamental solutions for the system (1) developed previously [Ricerche Mat. **7** (1958), 153–185; Bull. Acad. Polon. Sci. Sér. Math. Astr. Phys. **6** (1958), 78–83; MR **21** #4300, 2123], the author defines the generalized integrals of Poisson-Weierstrass:

$$(2) \quad J_\alpha(X, t) = \int_E \sum_{\beta=1}^{N} \Gamma_{\alpha\beta}(X, t; Y, 0) f_\beta(Y)\, dY$$

and the generalized potential of space charge of density $\{\rho_\beta\}$:

$$(3) \quad V_\alpha(X, t) = \int_0^t \int_E \sum_{\beta=1}^{N} \Gamma_{\alpha\beta}(X, t; Y, \tau) \rho_\beta(Y, \tau)\, dY\, d\tau,$$

where $\Gamma_{\alpha\beta}$ is this fundamental solution matrix. It is shown that when the functions $f_\alpha(X)$ are continuous on E and grow no faster than an exponential, then the integrals (2) are absolutely convergent for $0 < t \leq T$, they may be differentiated m times under the integral sign with respect to the spatial variables for $m \leq M - 1$, and they have the property $\lim_{t \to 0} J_\alpha(X, t) = f_\alpha(X)$. They also satisfy the system (1) for $0 < t < T$, and their spatial derivatives are shown to satisfy a certain kind of Hölder condition. Analogous properties for the integrals (3) are also established, under appropriate conditions on the density functions. The integrals (3) satisfy the inhomogeneous system $\hat{\Psi}^{(\alpha)}(V_1, \cdots, V_N) = -\rho_\alpha(X, t)$.

In the second portion of the paper, the above results are used in establishing an existence theorem for the Cauchy problem for a quasi-linear parabolic system of the form

$$\hat{\Psi}^{(\alpha)}(u) = F_\alpha(X, t, u, Du, D^2u, \cdots, D^{M-1}u) \qquad (\alpha = 1, \cdots, N),$$

where $D^\beta u$ represents the matrix of all the spatial derivatives of a fixed order β of the functions $u = (u_1, \cdots, u_N)$. The method involves recasting the problem in the form of a system of integral equations and using the fixed-point theorem of Schauder.

D. H. Hyers (Los Angeles, Calif.)

7362:

Friedman, Avner. Interior estimates for parabolic systems of partial differential equations. J. Math. Mech. **7** (1958), 393–417.

A system of linear equations of the form

$$\sum_{\mu=1}^{N} \sum_{0 \leq \sum i_j \leq m} a_{i_1 \cdots i_n}^{\lambda\mu}(x, t) \frac{\partial^l u_\mu}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} - \frac{du_\lambda}{\partial t} = f_\lambda(x, t)$$

for N functions of $n+1$ variables $(x, t) = (x_1, \cdots, x_n, t)$ is called "parabolic" if the real parts of the characteristic roots of the matrix

$$((-i)^m \sum_{\sum i_j = m} a_{i_1 \cdots i_n}^{\lambda\mu}(x, t) \xi_1^{i_1} \cdots \xi_n^{i_n}) \quad (\xi \text{ real}, |\xi| = 1)$$

are negative. In this paper, a priori estimates of Schauder's type are obtained pertaining to any solution (u_λ) of such a system defined in a finite, $(n+1)$-dimensional domain D. Consider any subset D_0 of D removed from the boundary of D by a positive "distance" δ, the "distance" between two points (x, t) and (x', t') being measured in this paper by $[|x - x'|^2 + (t - t')^{2/m}]^{1/2}$ ($|x - x'|$ denotes the Euclidean distance between x and x'). In D_0, $\partial u_\lambda/\partial t$, the x derivatives of the u_λ of orders up to m, and their moduli of continuity (of Hölder's type), are estimated from δ, $\sup_D \sum_{\lambda=1}^{N} |u_\lambda|$, and a bound and a modulus of continuity for f. The method, in brief, is to use S. D. Èidel'man's fundamental solution [Mat. Sb. (N.S.) **38** (**80**) (1956), 51–92; MR **17**, 857] in a procedure modelled on that of Douglis and Nirenberg [Comm. Pure Appl. Math. **8** (1955), 503–538; MR **17**, 743], the results generalizing analogous statements concerning second order equations of R. B. Barrar [unpublished thesis, Univ. of Mich.,

1952]. These results are applied to the question of differentiability of solutions of linear and non-linear systems of parabolic type. *A. Douglis* (College Park, Md.)

7363:

Friedman, Avner. Boundary estimates for second order parabolic equations and their applications. J. Math. Mech. **7** (1958), 771–791.

Consider a sufficiently regular, second order parabolic equation

$$(*)\quad \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} +c(x,t)-\frac{\partial u}{\partial t}=f(x,t)$$

in a finite $(n+1)$-dimensional domain D bounded by two planes $t=0$, $t=t_1>0$, and a surface C between them. A neighborhood of each point of C is supposed to be representable for some index i $(1\leqq i\leqq n)$ as $x_i=h(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n,t)$, where h is sufficiently smooth. Let u denote any sufficiently smooth solution of (*) in the closure $\bar{D}$ of D, and let ψ be any similarly smooth function in $\bar{D}$ such that $u=\psi$ on the part of the boundary of D not contained in the plane $t=t_1$. R. B. Barrar [unpublished thesis, Univ. of Mich., 1952] has shown that u, u_t, the first and second derivatives of u with respect to x, and also a modulus of continuity (of Hölder's type) for each of these functions, can be estimated in D from bounds and similar continuity moduli for f, ψ, and the first and second derivatives of ψ with respect to x. In this paper, a simpler derivation of these boundary estimates, based on the discussion of the first boundary problem for second order elliptic equations in Douglis and Nirenberg, Comm. Pure Appl. Math. **8** (1955), 503–538 [MR **17**, 743], is presented as a possible model for analogous studies of general parabolic systems. (Interior estimates for general parabolic systems have been given in the paper reviewed above.) The boundary estimates are then applied by means of a continuity argument to existence theorems for linear and non-linear parabolic equations.

A. Douglis (College Park, Md.)

7364:

Agranovič, M. S. Some theorems on partial differential equations with constant coefficients. Dokl. Akad. Nauk SSSR **128** (1959), 439–442. (Russian)

Let P be a differential operator in R^n with constant coefficients of order m, and let Ω and Ω' be open sets in R^n such that $\Omega'\supset\bar{\Omega}$. Let E be an elementary solution of P and suppose it is of order at most N. It is announced that $u\in C^{p+m+N}$ in $\Omega'-\Omega$ and $Pu\in C^{p+N}$ in Ω' imply $u\in C^p$ in Ω provided Ω' is bounded. Suitably interpreted, this holds also for $p<0$. {Reviewer's remark: This holds also for Ω not necessarily bounded but strictly convex [Malgrange, to be published].} Some refinements are given under the assumption that E vanishes in a half-space. Another result is that if $Pu=0$ in Ω' and u has finite order then $u=\sum(\partial^k/\partial x^k)u_k$ (finite sense) with $u_k\in C^l$ (l given), and $Pu_k=0$ in Ω. Finally, $Pu=f\in C^\infty$ is solved with $u\in C^\infty$ by an explicit construction using Hörmander's ladder. Analogous formulas can be made for f and u in other spaces. No proofs. *L. Gårding* (Lund)

7365:

Browder, Felix E. On the Dirichlet problem for linear non-elliptic partial differential equations. II. Rend. Circ. Mat. Palermo (2) **7** (1958), 303–308.

Let A be an arbitrary differential operator of order r, not necessarily elliptic, with smooth and bounded coefficients in an open set G of real n-space. Let $(f,g)_m$ be the scalar product in the space H_m of all functions with square integrable derivatives of order $\leqq m$, vanishing of order $m-1$ at the boundary of G. The subject of this paper is the Dirichlet problem $Au=f\in H_0$ with $u-g\in H_m$ (f and g given). The formula $(A\varphi,\Psi)_0=(S\varphi,\Psi)_m$ $(\varphi\in H_\infty;\Psi,S\varphi\in H_m)$ defines an operator $S:H_m\to H_m$. Define $h\in H_m$ by $(f,\varphi)_0-(Ag,\varphi)_0=(h,\varphi)_m$. Then provided $h=Tk$ for some $k\in H_m$ and some extension T of S, Dirichlet's problem has the solution $u=g+k$.

Assuming that $T=T_1+iT_2$ with T closed, T_1 self-adjoint and T_2 symmetric, both with the domain of T, the author gives a sufficient condition for the $h\in H_m$ to be in the domain of T. The paper simplifies and extends earlier papers by the author [same Rend. **6** (1957), 249–253; MR **21** #2819], Louhivaara [same Rend. (2) **5** (1956), 260–274; MR **19**, 281], and Littman [Comm. Pure Appl. Math. **11** (1958), 145–151; MR **20** #3382].

L. Gårding (Lund)

7366:

Kalašnikov, A. S. Construction of generalized solutions of quasi-linear equations of first order without convexity conditions as limits of solutions of parabolic equations with a small parameter. Dokl. Akad. Nauk SSSR **127** (1959), 27–30. (Russian)

The author investigates Cauchy's problem

$$(1)\quad \frac{\partial u}{\partial t}+\frac{\partial\varphi(u)}{\partial x}=0 \quad\text{in}\quad S(-\infty<x<\infty,\ 0\leqq t\leqq T),$$

$$(2)\qquad u(0,x)=u_0(x)\quad(-\infty<x<\infty).$$

He defines a so-called generalised solution of this problem. According to O. A. Oleĭnik's works such a solution exists and is unique for any measurable function $u_0(x)$ if $\varphi''(u)\neq 0$; the solution may be expressed as a limit of the solution of the problem

$$\varepsilon\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\frac{\partial\varphi(u)}{\partial x},\quad u(0,x)=u_0(x)$$

for $\varepsilon\to 0$.

In this paper, the author proves—under some assumptions about the function $u_0(x)$—that this result holds also for the case when $\varphi''(u)$ has a finite number of zero-points in any finite interval for u.

K. Rektorys (Prague)

POTENTIAL THEORY

7367:

Johnson, Guy. Functions which have harmonic support. Trans. Amer. Math. Soc. **92** (1959), 302–321.

L'A. étudie dans un domaine plan D les fonctions sousharmoniques u (dites de classe (hs)) telles que (a) u est enveloppe supérieure d'une famille de fonctions harmoniques localement bornées supérieurement; (b) ou, ce qui

est équivalent, u majore pour chaque point $x_0 \in D$, une fonction harmonique dans D, égale à u en x_0 (ce qui implique la continuité de u). Quelques propriétés immédiates: $-u$, s'il est >0, satisfait aux inégalités de Harnack, et la limite d'une suite décroissante (donc d'un ordonné filtrant décroissant) des fonctions de (hs) est $-\infty$ ou une fonction de (hs). La convergence uniforme sur tout compact donne aussi une fonction de (hs) et toute fonction de (hs) est limite avec convergence uniforme sur tout compact d'une suite de fonctions de (hs), indéfiniment dérivables.

Le théorème "essentiel" est qu'une famille uniformément bornée supérieurement localement de fonctions de (hs) est "normale". L'A. se base sur une majoration pénible des masses associées, alors que c'est une conséquence immédiate des inégalités étendus de Harnack (qui entraînent par exemple l'égale continuité en un point des e^u pour les $u<0$ au voisinage).

Enfin on signale comme sous-classe intéressante celle des fonctions $v \geqq 0$ telles que $\log v \in (hs)$, car cela équivaut, si D est simplement connexe, à ce que v est l'enveloppe supérieure des modules des fonctions holomorphes non nulles d'une famille. *M. Brelot* (Paris)

7368:

Ullman, J. L. On Tchebycheff polynomials. Pacific J. Math. **9** (1959), 913–923.

Let C be a closed bounded set in the z-plane, which is of positive logarithmic capacity; let T_n be the Tchebycheff polynomial of degree n associated with C and U the conductor potential of C: $U(z)=\int \log|z-\zeta|d\mu_\zeta$; let $\nu_n(S)$ denote the total multiplicity of the zeros of T_n on S. Under the hypothesis that the complement D of C is connected, the author proves the following. (1) For any closed subset E of D,

$$\nu_n(E) = o(n), \quad \int_E |n^{-1}\log|T_n| - U|dxdy \to 0$$

as $n\to\infty$. (2) For any smooth Jordan curve Γ in D, with the interior I, $\nu_n(I)/n \to \mu(I)$, as $n\to\infty$, where μ is the mass-distribution associated with U. (3) If C is the union of a finite number of disjoint analytic Jordan curves and E a closed subset of D, $\nu_n(E)$ is bounded by a constant depending only on E, but not on n.

$$\lim_{n\to\infty} n^{-1}\log|T_n(z)| = U(z) \tag{4}$$

for z in D, with the possible exception of a set of measure zero. *A. Pfluger* (Zürich)

7369:

Gluško, V. P. On operators of potential type and some imbedding theorems. Dokl. Akad. Nauk SSSR **126** (1959), 467–470. (Russian)

The author is interested in potential operators of the type

$$A_\lambda\varphi(P) = \int_\Omega \varphi(Q)r^{-\lambda}(P,Q)dQ,$$

where $r(P,Q)$ represents the distance in n-dimensional Euclidean space. A representative situation is the following: Let Ω be a domain in n-dimensional Euclidean space R_n and Ω_s the intersection of Ω with the subspace $x_{s+1}=x_{s+2}=\cdots=x_n=0$. Further, let $L_{p,h(M)}$ and $L^s_{p,h(M)}$ be the Banach spaces of equivalence classes of functions which satisfy

$$\int_\Omega |\varphi(Q)|^p r^{-kp}(M,Q)dQ < \infty$$

and

$$\int_{\Omega_s} |\varphi(Q)|^p r^{-kp}(M,Q)dQ < \infty,$$

respectively, where M is a fixed point in R_n. If $p\geqq 1$, $s\leqq n$, $-n/p' < k < s/p$ $(1/p+1/p'=1)$, then the operator A_λ for $(n/p')+k<\lambda<(n/p')+(s/p)$ acts as a bounded operator from $L_{p,h(\Omega)}$ to $L^s_{q,h(\Omega)}$, where h is any number satisfying the inequality $k<h\leqq k+(n/p')+(s/p)-\lambda$, and $q=sp[n-(n-\lambda-h+k)p]^{-1}$. For allied results see Gluško and Krein [Dokl. Acad. Nauk SSSR **122** (1958), 963–966; MR **20** #6578]. *A. Devinatz* (St. Louis, Mo.)

7370:

Szybiak, A. On some constants related to the generalized potentials. Ann. Polon. Math. **6** (1959), 265–268.

Let $K(x)$ be a function which is continuous in $\mathcal{E}^m$ except perhaps at $x=0$ and which satisfies the following conditions: (i) $0<\lim_{x\to 0}K(x)\leqq\infty$; (ii) $K(x)=K(-x)$; (iii) $\int_{|x|<1}K(x)dx<\infty$; (iv) for each measure μ with carrier $F\subset\mathcal{E}^m$

$$\sup_{x\in\mathcal{E}^m}\int K(x-y)d\mu(y) = \sup_{x\in F}\int K(x-y)d\mu(y).$$

The function K is called the kernel of the generalized potential $\int K(x-y)d\mu(y)$. Next let E be a compact set in $\mathcal{E}^m$ for which

$$\gamma_E = \inf_\mu \iint K(x-y)d\mu(y)d\mu(x) < \infty,$$

where $\mu\geqq 0$, $\mu(E)=1$ and $\mu(\mathcal{E}^m-E)=0$. The author gives two alternative characterizations for γ_E. For example, he proves that

$$\gamma_E = \lim_{n\to\infty}\left((n+1)^{-2}\inf_{(x_i)}\sum_{i\neq j}K(x_i-x_j)\right),$$

where for each n, the infimum is taken over all systems of $n+1$ points $\{x_0,\cdots,x_n\}\subset E$. *F. W. Gehring* (Zürich)

7371:

★Frostman, Otto. Suites convergentes de distributions d'équilibre. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 86–89. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

Let F be a compact set in a euclidean space and let $\alpha>0$. The energy $I^{(\alpha)}(\mu)$ of a measure μ is defined by $\iint |x-y|^{-\alpha}d\mu(x)d\mu(y)$ and the capacity $C^{(\alpha)}(F)$ is defined by $\inf 1/I^{(\alpha)}(\mu)$ taken with respect to all unit non-negative mass-distributions supported by F. It is known that there is such a measure $\mu^{(\alpha)}$ which minimizes the energy and that this is unique if $C^{(\alpha)}(F)>0$. Given $\beta>0$, assume that $C^{(\alpha)}(F)>0$ for every $\alpha<\beta$, and consider $\{\mu^{(\alpha)}\}$. The problem is to study the behavior of $\mu^{(\alpha)}$ as $\alpha\to\beta$. In case $C^{(\beta)}(F)>0$, it is easily seen that $\mu^{(\alpha)}$ converges vaguely to $\mu^{(\beta)}$ as $\alpha\to\beta$. If $C^{(\beta)}(F)=0$, the following result is obtained in the special case when F is a rectifiable curve of

length l in a plane: $\lim_{\alpha\to 1} C^{(\alpha)}(F)(1-\alpha)^{-1} = l/2$ and $\mu^{(\alpha)}$ converges vaguely to a certain mass-distribution on F.

{The proof is not complete as it stands but can be corrected by considering a subset, with linear measure arbitrarily close to l, of F where uniformly

$$\limsup_{r\to 0} s(x, r)r^{-1} \leqq 2.$$

This correction has been written through correspondence with the author.} *M. Ohtsuka* (Hiroshima)

7372:

Kishi, Masanori. On the capacitability of analytic sets. Proc. Japan Acad. **35** (1959), 158–160.

The author continues his study on the capacitability; see Nagoya Math. J. **12** (1957), 195–219 [MR **20** #2550] for his previous work. Let Ω be a locally compact separable metric space, and Φ be a positive symmetric kernel which satisfies the continuity principle and has the following property: when Ω is not compact, there exists, for any compact set K and for any $\varepsilon > 0$, a compact set $L \supset K$ such that $\Phi(P, Q) < \varepsilon$ in $K \times (\Omega - L)$. Let μ be a positive measure whose total mass is finite. Every subset of a set on which U^μ is infinite is called a polar set and the family of all polar sets is denoted by $\mathfrak{P}$. For an arbitrary set X, let $\mathfrak{F}_X$ denote the set $\{\mu > 0;\ U^\mu \geqq 1$ on X except $E \in \mathfrak{P}\}$. The inner capacity $\mathfrak{F}\text{-cap}_i(X)$ is defined by $\sup \inf \mu(\Omega)$, taken with respect to $\mu \in \mathfrak{F}_K$ and $K \subset X$, and the outer capacity by $\inf \mathfrak{F}\text{-cap}_i(G)$ taken with respect to open sets $G \supset X$. The main theorem asserts that they coincide for every analytic set K.

M. Ohtsuka (Hiroshima)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 7515.

7373:

Abdel-Messih, M. A. A Green's function analogue for ordinary linear difference equations. Proc. Math. Phys. Soc. Egypt. No. 22 (1958), 43–51 (1959). (Arabic summary)

The author defines a function $G(x, z)$ by means of which the solution of the non-homogeneous linear difference equation

$$M(u(x)) \equiv a(x)u(x-h) + b(x)u(x) + c(x)u(x-h) = -\phi(x)$$

with given homogeneous boundary conditions on the interval $x_0 \leqq x \leqq x_m$, is obtained in the form of the sum

$$u(x) = \sum_{z=z_1}^{z_{m-1}} G(x, z)\phi(z).$$

The function $G(x, z)$ is called a Green's function for the difference equation because of the analogy between its role and that of the Green's function in the theory of differential equations. The Green's function is constructed for the second order equation with constant coefficients and for the second order equation with variable coefficients having suitable boundary conditions. The extension of the method to equations of higher order is indicated for an analogous fourth order equation. As a numerical example the exact solution of the differential equation $u'' + u = -x$, $u(0) = u(1) = 0$, is compared with the solution of the related approximating difference equation obtained by the method of the author.

P. E. Guenther (Cleveland, Ohio)

7374:

Aczél, Johann. Miszellen über Funktionalgleichungen. I. Math. Nachr. **19** (1958), 87–99.

The author considers a number of functional equations and systems of functional equations, in each case tracing the evolution of the particular problem through a sequence of works by earlier contributors, and ending with his own extension of the problem. From the variety of problems and results discussed by the author, the following two theorems are selected as representative.

(1) If the equation $f(x+y) = F(f(x), f(y))$, where x and y belong to a real interval (A, B), has a continuous, strictly monotone solution $f_0(x)$, the most general solution which is majorisable on a set of positive measure by a measurable function is of the form $f(x) = f_0(ax)$. Each such solution is continuous and, if not identically constant, strictly monotone.

(2) The system of functional equations

$$\begin{aligned} c(x+y) &= c(x)c(y) + s(x)s(y), \\ s(x+y) &= c(x)s(y) + s(x)c(y), \\ e(x+y) &= e(x)e(y), \quad e(x) \not\equiv 0, \\ c(x)^2 &= s(x)^2 + e(x)^2, \end{aligned}$$

where $c(x)$, $s(x)$, $e(x)$ are continuous complex functions of the complex variable x, has for its general solution the set of functions

$$e(x) = \exp(Cx + D\bar{x}), \quad c(x) = e(x)\operatorname{ch}(Ex + F\bar{x}),$$
$$s(x) = e(x)\operatorname{sh}(Ex + F\bar{x})$$

where C, D, E, F are arbitrary constants.

P. E. Guenther (Cleveland, Ohio)

7375:

Erdős, Jenő. A remark on the paper "On some functional equations" by S. Kurepa. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 3–5. (Serbo-Croatian summary)

Verf. betrachtet die von S. Kurepa in demselben Glasnik **11** (1956), 3–5 [MR **18**, 217] unter Derivierbarkeitsveraussetzung gelöste Funktionalgleichung

$$(1) \qquad f(x+y, z) + f(x, y) = f(y, z) + f(x, y+z),$$

und beweist unter Verwendung der Schreierschen Gruppenerweiterungstheorie, insbes. eines Satzes von R. Baer [vgl. z.B. A. G. Kurosch, *Gruppentheorie*, Berlin, 1953; MR **15**, 681; in der Arbeit selbst werden keine Literaturhinweise gegeben], dass jede symmetrische Lösung von (1) von der Gestalt

$$(2) \qquad f(x, y) = g(x+y) - g(x) - g(y)$$

ist. Umgekehrt erfüllt natürlich jede Funktion der Gestalt (2) diese Funktionalgleichung (1) und ist symmetrisch.

Verf. weist dann darauf hin, dass es durch gruppentheoretische Betrachtungen bewiesen werden kann, dass jede stetige Lösung von (2) symmetrisch (also von der Gestalt (2)) ist, er gibt aber in extenso einem vom Ref.

gefundenen analytischen Beweis dieser Behauptung durch Zurückführung auf die Cauchysche Funktionalgleichung wieder, was zugleich zeigt, dass die Stetigkeitsbedingung gemildert werden kann.

Endlich konstruiert der Verf. mittels der Hamelschen Basis [Math. Ann. **60** (1905), 459–462] eine (nichtstetige) Lösung von (1), die nicht von der Gestalt (2) ist.

J. Aczél (Debrecen)

7376:

Gheorghiu, Oct. Em.; et Mioc, V. Sur un système d'équations fonctionnelles. Com. Acad. R. P. Romîne **8** (1958), 755–763. (Romanian. Russian and French summaries)

The authors study a very general system of functional equations by means of a generalization of the cosines of P. Appell [C. R. Acad. Sci. Paris **84** (1877), 540–543, 1378–1380] and a special hypercomplex number system whose basis is $\theta, \theta^2, \cdots, \theta^n = 1$. Solution of a certain special case is obtained in terms of arbitrary constants.

J. A. Ward (Holloman Air Force Base, N.M.)

SEQUENCES, SERIES, SUMMABILITY

See also 7136.

7377:

Myrberg, P. J. Iteration von Quadratwurzeloperationen. Ann. Acad. Sci. Fenn. Ser. A. I. no. 259 (1958), 16 pp.

Es wird untersucht, wann eine Entwicklung

$$(p) = \pm\sqrt{(p \pm \sqrt{(p \pm \sqrt{(p \pm \cdots)})}}, \quad p > 0,$$

mit periodischer Vorzeichenverteilung konvergiert. Dies Problem wird zurückgeführt auf das Studium der Fixpunkte von iterierten quadratischen Polynomen, die vom Verfasser in einer früheren Arbeit [dieselben Ann. no. 256 (1958); MR **20** #5876] behandelt wurden. Das prägnanteste Resultat besagt, daß (p) für $p \geq 2$ stets konvergiert. Die Arbeit schließt mit der Behandlung einiger Spezialfälle.

H. Tietz (Münster)

7378:

Hirokawa, Hiroshi. Riemann-Cesàro methods of summability. III. Tôhoku Math. J. (2) **11** (1959), 130–145.

[Pour la première et la seconde partie de cette note voir le même J. **7** (1955), 279–295; **9** (1957), 13–26; MR **17**, 1076; **19**, 851.] Avec la définition et les notations de sommabilité de Riemann-Cesàro (R, p, α) d'ordre p et d'indice α, données aux I et II, les résultats de l'auteur s'expriment comme suit. Il existe des séries sommables-$(R, 2, \alpha+1)$, $-1 \leq \alpha < 0$, mais pas sommables-$(R, 2, \alpha)$, ainsi que des séries qui sont sommables-$(R, 2, \alpha)$, $-1 \leq \alpha < 0$, mais qui ne sont pas sommables-$(R, 2, \alpha+1)$. Dans le cas de sommabilité de Riemann R_2 et $(R, 2)$ qui ne sont qu'un cas spécial de (R, p, α), les résultats analogues, c.à.d. que R_2 et $(R, 2)$ ne sont pas comparables, sont donnés par J. Marcinkiewicz [J. London Math. Soc. **40** (1936), 268–272] et par B. Kuttner [Proc. London Math. Soc. **40** (1936), 524–540]. Le reste du note est consacré aux études des relations entre les sommabilités d'Abel et de $|C, p|$ d'une part et la sommabilité (R, p, α) d'autre part.

M. Tomić (Belgrade)

7379:

Hirokawa, Hiroshi. Riemann-Cesàro methods of summability. IV. Tôhoku Math. J. (2) **11** (1959), 271–286.

C. T. Rajagopal [même J. **9** (1957), 247–263; **10** (1958), 366; MR **20** #1871] a défini le procédé de sommation de Riemann-Cesàro (R, p, α) pour les intégrales. L'auteur donne ici les analogues intégrales de ses théorèmes de III [voir ci-dessus], généralisant en même temps quelques résultats de Rajagopal. Par exemple, de A-, (C, r)-, ou bien $|C, p|$-limite de

$$s_\alpha(u) = (\Gamma(\alpha+1))^{-1} \int_0^u (u-x)^\alpha a(x)dx$$

résulte (R, p, α)-limite de cette intégrale sous des conditions supplémentaires imposées sur $\sigma_r(x)$ et $s_\alpha(u)$, $0 < r+1 \leq \alpha+1 < p$.

M. Tomić (*Belgrade*)

7380:

Agnew, R. P.; and Fuchs, W. H. J. Inclusion relations among absolute Abel summation methods. Scripta Math. **24** (1959), 133–136.

A series $\sum a_n$ is said to be summable $|A|_r$ ($r > 0$, not necessarily an integer) if the series in $f(x) = \sum a_n x^n$ converges for $0 \leq x < 1$ and the integral

$$\text{(A)} \qquad I_r = I_r(f') = \int_0^1 (1-x)^{r-1} |f'(x)|^r dx < \infty.$$

Concerning the significance of the integral (A) in the theory of functions, investigations have been made by T. M. Flett [Proc. London Math. Soc. (3) **7** (1957), 113–141; MR **19**, 266], D. Waterman [Trans. Amer. Math. Soc. **69** (1950), 468–478; MR **12**, 496] and others. In this paper, the authors give two fundamental inclusion relations for the method of summation $|A|_r$. They prove that, if $0 < a < b$, and $I_r < \infty$ for $r = a$ and $r = b$, then $I_r < \infty$ for every $a \leq r \leq b$; and that, if $0 < a < b$, then there are functions $f(z)$, analytic over the disk $|z| < 1$, for which $I_r < \infty$ if and only if $a \leq r \leq b$.

Fu Cheng Hsiang (Taipeh)

7381:

Borwein, D.; and Boyd, A. V. Binary and ternary transformations of sequences. Proc. Edinburgh Math. Soc. **11** (1958/59), 175–181.

The sequence $\{s_n\}$ is said to be summable $T^r(\alpha)$ to s if the sequence $\{s_n(r, \alpha)\}$,

$$s_n(r, \alpha) = (q+1)^{-r} \sum_{k=n-r}^{n} \binom{r}{n-k} q^{n-k} s_k$$

(where $q = 1/(\alpha-1)$, $\alpha \neq 0$), converges to s. It is said to be summable $T^\infty(\alpha)$ to s if $\{s_n(n, \alpha)\}$ converges to s. That $T^\infty(\alpha)$ is equivalent to the Euler method (E, q) [in the notation of Hardy, *Divergent series*, Clarendon Press, Oxford, 1949; MR **11**, 25] is evident. Comparing the $T^r(\alpha)$ methods for varying r and α, the authors prove that (i) for $r \geq 0$, $T^{r+1}(\alpha) \supset T^r(\alpha)$ for any α, and $T^\infty(\alpha) \supset T^r(\alpha)$ if and only if $0 < \alpha \leq 1$; (ii) for $\alpha, \beta \leq \frac{1}{2}$, $\alpha \neq \beta$, $T^r(\alpha)$ and $T^s(\beta)$ cannot be equivalent for any α, β, r and s. Writing the $T^r(\alpha)$ transformation as a suitable Nörlund transformation and using various known results on the Nörlund methods, they prove that (i) for $s > 0$, summability by the Cesàro method (C, s) cannot imply that by $T^r(\alpha)$; (ii) for

$\alpha > \frac{1}{2}$, $T^r(\alpha)$ is equivalent to convergence; (iii) $(C, r) \supset T^r(\frac{1}{2})$; and (iv) for $\alpha < \frac{1}{2}$ and $\alpha \neq 0$, $T^r(\alpha)$ does not imply Abel summability.

Similar comparisons are made between the Cesàro or Abel methods and the $T(\alpha, \beta)$ transforms which carry $\{s_n\}$ into $\{s_n'\}$ where $s_0' = \alpha s_0$, $s_1' = \alpha s_1 + \beta s_0$ and $s_n' = \alpha s_n + \beta s_{n-1} + (1-\alpha-\beta)s_{n-2}$ $(n \geq 2)$.

M. S. Ramanujan (Ann Arbor, Mich.)

7382:

Alexiewicz, A.; and Orlicz, W. Consistency theorems for Banach space analogues of Toeplitzian methods of summability. Studia Math. **18** (1959), 199–210.

Let X and Y be two Banach spaces and $\{A_\nu\}$ a system of linear operators from X to Y. If the transforms $A_i(x) = \sum_{\nu=0}^{\infty} A_{i\nu}(x_\nu)$, $x = \{x_\nu\} \in X$, be such that all the series $A_i(x)$ are convergent [or weakly convergent] and $\lim_{i\to\infty} A_i(x) = A_\cdot(x)$ [or $A_\cdot(x) = \text{w-lim } A_i(x)$], then the sequence x is said to be strongly [or weakly] A-summable to $A_\cdot(x)$, or, briefly, s-A-summable [or w-A-summable] to $A_\cdot(x)$. If the series $A_i(x)$ are weakly convergent and $A_i(x)$ converges strongly, x is said to be m-A-summable. $\mathfrak{M}$, $\mathfrak{C}$, $\mathfrak{C}^0$ denote, respectively, the classes of bounded, convergent and null sequences in X. $\mathfrak{A}_0^k$, $\mathfrak{A}^k$ denote the classes of sequences x which are k-A-summable to 0 and k-A-summable, where k stands for s, w or m. In section 1, the authors state several necessary and sufficient conditions for A to be k-conservative or k-permanent; these conditions are analogous to those of Robinson [Proc. London Math. Soc. (2) **52** (1950), 132–160; MR **12**, 253] and Melvin-Melvin [ibid. **53** (1951), 83–108; MR **13**, 45].

Section 2 is devoted to the linear spaces $\mathfrak{M} \cap \mathfrak{A}_0^s$ and $\mathfrak{M} \cap \mathfrak{A}_0^m$, each of which is a two-norm space; the norms of the first are

$$\|x\| = \sup_\nu \|x_\nu\|,$$

$$\|x\|^* = \sum_{\nu=0}^{\infty} 2^{-\nu}\Big(\|x_\nu\| + \sup_\mu \Big\|\sum_{\sigma=0}^{\mu} A_{\nu\sigma}(x_\sigma)\Big\|\Big) + \sup_\nu \|A_\nu(x)\|,$$

while those of the second are

$$\|x\| = \sup_\nu \|x_\nu\|,$$

$$\|x\|^* = \sum_{\nu=0}^{\infty} 2^{-\nu}\|x_\nu\| + \sup_\nu \|A_\nu(x)\|.$$

In a linear space Z with two norms $\|\cdot\|$ and $\|\cdot\|^*$, denoted by $\langle Z, \|\cdot\|, \|\cdot\|^*\rangle$, a sequence $\{z_n\}$ is called γ-convergent to z if $\sup_n \|z_n\| < \infty$ and $\lim_n \|z_n - z\|^* = 0$. The terms γ-complete, γ-dense and γ-linear are analogously defined. The authors prove that the spaces $\langle \mathfrak{M} \cap \mathfrak{A}_0^s, \|\cdot\|, \|\cdot\|^*\rangle$ and $\langle \mathfrak{M} \cap \mathfrak{A}_0^m, \|\cdot\|, \|\cdot\|^*\rangle$ are γ-complete, that in these spaces the limit of each pointwise convergent sequence of γ-linear functionals is γ-linear, and that the set $\mathfrak{C}_0^*$ composed of sequences with only a finite number of non-zero terms is γ-dense in both spaces. Finally they prove a set of consistency theorems, typical of which is the following. Let the methods $A^{(p)}$ $(p = 0, 1, \cdots)$ be s-null-permanent, and let B be w-null-permanent. Let k = s or k = m; if every bounded sequence x which is k-$A^{(p)}$-summable to zero for $p = 0, 1, \cdots$ is w-B-summable, then $B_\cdot(x) = 0$.

M. S. Ramanujan (Ann Arbor, Mich.)

7383:

★Ganelius, Tord. General and special Tauberian remainder theorems. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 102–103. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

Suppose that $K(x) \in L$ and $\hat{K}(t) = \int e^{-itx}K(x)dx$ is a regular analytic function in the strip $|\text{Im } t| < a$, and that there are constants H, h so that $|\hat{K}(t)| \leq He^{h|t|}$ in the strip. Then, if $\Phi(x)$ is bounded and $\Phi(x) + Cx$ is non-decreasing for some constant C,

$$\int K(x-y)\Phi(y)dy = O(e^{-\gamma x}) \quad (\gamma > 0)$$

implies that $\Phi(x) = O(x^{-1})$ as $x \to \infty$. In particular, $K(t)$ may be the Abel kernel $\exp(-x - e^{-x})$. The results are related to some due to Beurling [9th Congr. Math. Scand., Helsinki, 1938, pp. 345–366, Mercators, Helsinki, 1939], but these need the stronger condition that $|\hat{K}(t)|^{-1} = O(t^n)$.

H. R. Pitt (Nottingham)

7384:

Srivastava, Pramila. A Tauberian theorem for strong Riesz summability. Proc. Amer. Math. Soc. **10** (1959), 540–544.

The series $\sum a_n$ is summable (R, λ, k) to s if $x^{-k}A_\lambda^k(x) \to s$ as $x \to \infty$, where $A_\lambda^k(x) = \sum_{\lambda_n < x} (x - \lambda_n)^k a_n$ and the numbers λ_n are positive, increasing and unbounded. The series is summable $[R, \lambda, k]$ if $\int_{\lambda_0}^{\omega} |x^{-k+1}A_\lambda^{k-1}(x) - s|dx = o(\omega)$ as $\omega \to \infty$, and is bounded $[R, \lambda, k]$ if the last integral is $O(\omega)$. It is proved that if $\sum a_n$ is summable (R, λ, k) for some positive k and is bounded $[R, \lambda, r]$ for some $r \geq 0$, then it is summable $[R, \lambda, r + \delta]$ for every positive δ. The case $\lambda_n = n$, in which the Riesz means $A_\lambda^k(x)$ become Cesàro means, has been proved by C. E. Winn [Math. Z. **37** (1933), 481–492].

H. R. Pitt (Nottingham)

APPROXIMATIONS AND EXPANSIONS

7385:

Curtis, Philip C., Jr. n-parameter families and best approximation. Pacific J. Math. **9** (1959), 1013–1027.

An n-parameter family F on a closed finite interval $[a, b]$ is a family of single-valued real continuous functions $f(x)$ on $[a, b]$ such that for every set of points (x_i, y_i), $i = 1, \cdots, n$, with $a \leq x_1 < x_2 < \cdots < x_n \leq b$ there is exactly one $f(x)$ in F with $f(x_i) = y_i$. F may but need not be linear. Considering F as a class of approximating functions, a best approximant in F to a continuous function g, relative to the norm $\|\ \|$, is a function $f \in F$ for which $\|f - g\| = \inf_{f \in F} \|f - g\|$. Adopting the norm $\|f\| = \max_{a \leq x \leq b} |f(x)|$, several results due to Motzkin [Bull. Amer. Math. Soc. **55** (1949), 789–793; MR **11**, 101], Tornheim [Trans. Amer. Math. Soc. **69** (1950), 457–467; MR **12**, 395] and Novodvorskiĭ and Pinsker [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 6 (46), 174–181; MR **13**, 728] concerning the unique existence and characterization of f are proved and cleared from redundant assumptions. Two iterative procedures for constructing f are given and shown to converge. Adopting the $L_{p,N}$ norm $\|f\| = (\sum_{i=1}^{N} |f(x_i)|^p)^{1/p}$, where $a \leq x_1 < x_2 < \cdots < x_N \leq b$, $N \geq n$, $p > 1$, it is shown that F has a unique best approximant to any continuous function if and only if F is the translate of a linear n-parameter family. The problem of the existence of n-parameter families F on general compact spaces S is also discussed. It is shown, under additional hypotheses on F, that S

must be homeomorphic to a subset of the circumference of the unit circle. If n is even, this subset must be proper. This extends a result proved earlier for linear n-parameter families by Mairhuber [Proc. Amer. Math. Soc. **7** (1956), 609–615; MR **18**, 125].

Walter Gautschi (Oak Ridge, Tenn.)

7386:

Gupta, D. P. Correction: On order of sum of the series of ultraspherical functions. Tôhoku Math. J. (2) **11** (1959), 489.

Original is in same J. **10** (1958), 42–48 [MR **20** #1876].

7387:

Meder, J. On the summability almost everywhere of orthonormal series by the method of first logarithmic means. Rozprawy Mat. **17**, 34 pp. (1959).

The author gives eleven theorems on the first Riesz logarithmic means of orthonormal series. These are strictly counterparts of theorems on the first arithmetic means of orthonormal series. But some of them are given in more general form by G. G. Lorentz [Trans. Roy. Soc. Canada. Sect. III (3) **45** (1951), 19–32; MR **14**, 160].

G. Sunouchi (Evanston, Ill.)

7388:

Motzkin, T. S.; and Walsh, J. L. Polynomials of best approximation on a real finite point set. I. Trans. Amer. Math. Soc. **91** (1959), 231–245.

Let E denote a finite real point set containing at least $n+2$ points and $f(x)$ a real function defined on E. An nth degree polynomial $p_n(x)$ is said to be a juxtapolynomial of $f(x)$ on E if there does not exist any closer nth degree polynomial $q_n(x)$ to $f(x)$ on E in the sense $|f(x)-q_n(x)| < |f(x)-p_n(x)|$ on E where $f(x)\neq p_n(x)$ but $f(x)-q_n(x)=f(x)-p_n(x)$ on E where $f(x)=p_n(x)$. In this paper juxtapolynomials are identified with polynomials of best approximation and those for which $[f(x)-p_n(x)]$ has a certain number of sign changes. Specifically, the following three classes of $p_n(x)$ are shown to be the same: (1) $[f(x)-p_n(x)]$ has at least $n+1$ weak sign changes on E; (2) $p_n(x)$ is a juxtapolynomial of $f(x)$ on E; (3) $p_n(x)$ is a polynomial of best approximation to $f(x)$ on E, in the sense of least first powers with positive weights. If strong sign changes are assumed instead and if $f(x)\neq$ any nth degree polynomial on E, the best approximation is in the sense of least pth powers with positive weights.

M. Marden (Madison, Wis.)

7389:

Riney, T. D. Coefficients in certain asymptotic factorial expansions. Proc. Amer. Math. Soc. **10** (1959), 511–518.

Consider the function

$$g(w) = \prod_{i=1}^{p} \Gamma(w+\sigma_i) / \prod_{j=1}^{q} \Gamma(w+\rho_j),$$

where p and q are integers with $0 \leq p \leq q$, and where σ_i and ρ_j are independent of w. For large $|w|$ in the sector $|\arg w| < \pi-\varepsilon$, the asymptotic behavior of $g(w)$ is related to a series of the form $\sum_{m=0}^{M} c_m/\Gamma(\alpha w+\beta+m)$ where α and β are easily obtained. Similarly the behavior of $(g(w))^{-1}$ is related to a series of the form $\sum_{m=0}^{M} \gamma_m \Gamma(\alpha w+\beta-m)$. Various formulas for finding c_m and γ_m are known. In the present paper there is presented a general method which yields a finite linear recursion formula of fixed length q for each of these coefficients.

T. E. Hull (Stanford, Calif.)

7390:

van der Corput, J. G. On the coefficients in certain asymptotic factorial expansions. III, IV, V, VI, VII. Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math. **21** (1959), 153–164, 329–360.

These papers continue the results of two earlier papers [parts I and II, same Indag. **19** (1957), 337–345; 346–351; MR **20** #5391]. A small error in part I is corrected. Asymptotic series of the general form

$$\sum_m a_m \Gamma(z+u)/\Gamma(z+u+m+s), \quad \sum_m a_m \Gamma(z+u-m-s)/\Gamma(z+u)$$

are introduced, and a large number of results connected with such expansions are given. The kind of result is indicated very briefly as follows. A necessary and sufficient condition is given for a function to possess such an asymptotic expansion for large $|z|$. Addition and multiplication theorems are given. The coefficients in the expansion of $f^p(z)$ are related to those in the expansion of $f(z)$, and other properties of these coefficients are obtained. A principle of duality is introduced. Simple formulas for numerical differentiation and integration are derived. Recurrence relations for the coefficients in certain expansions are studied. The asymptotic behavior of $f(z)$ is found from that of $f(z+\alpha)/f(z)$, and an example is worked out in detail. Conditions for recurrence relations to be of bounded length are obtained. The asymptotic behavior of the solutions of certain functional equations is also considered.

T. E. Hull (Stanford, Calif.)

7391:

van der Corput, J. G. Neutrices. J. Soc. Indust. Appl. Math. **7** (1959), 253–279.

In this expository article, the author describes an extension of the idea of numbers called a neutrix N, which is a triplet (N', N'', N''') where N' is a non-empty set (the domain), N'' is an additive group (the range) and N''' is an additive group of functions defined over N' with values in N'' and such that the only constant function is the null function (the negligible functions). If N''' is the additive group of functions of the form $\varphi(\xi)+c$, where φ belongs to N''' and c belongs to N'', then the mapping $N'''\to N'''/N''$ associates with each function of N''' an element of N''. This element is defined to be the value of the function at N and, if f is the functional symbol, this value is denoted by $f(N)$.

This concept is useful in situations where relations between functions are additive. Where multiplication of functions is concerned, difficulties arise.

The author gives 8 examples of applications of this idea to a variety of situations typical of classical analysis and particularly to asymptotic series. The first example defines the function $\sum_{n=1}^{N} 1/n$ for a particular neutrix N. The value of this function is γ (Euler's constant). Similarly for the same neutrix

$$\sum_{n=N}^{N^2} \frac{1}{n} = 0, \quad \sum_{n=2N}^{N^2} \frac{1}{n} = -\log 2.$$

The author observes that these formulae are not impossible

because the left-hand sides are not sums of positive terms, but even so it seems to the reviewer that these bizarre results indicate that the concept of a neutrix is one which is best applied in carefully defined and specific circumstances.

The difficulty of using neutrices lies in finding the appropriate definition of negligible functions, and it seems likely that they will be most useful in problems about which a good deal is already known or as an exposition of known facts. The author does not give rules for the choice of the class of negligible functions. In fact, he asserts that "the terms which for some reason annoy us are called negligible." *H. C. Eggleston* (London)

FOURIER ANALYSIS

See also 7410.

7392:

Goes, Günther. Komplementäre Fourier-koeffizientenräume und Multiplikatoren. Math. Ann. **137** (1959), 371–384.

In this paper the author continues his investigations on spaces of Fourier coefficients [cf. Math. Z. **70** (1959), 345–371; MR **21** #3711]. For the notations we refer to the review of the earlier paper. According to one of the definitions in that paper, the complementary subspace E^* of the linear subspace $E \subset P_\infty$ consists of all (a_n, b_n) such that $\sum_1^\infty (a_n c_n + b_n d_n)$ converges for all $(c_n, d_n) \in E$. In the present paper the Cesàro-complementary subspace $(C_1\text{-}E)^*$ of E is introduced; the convergence of $\sum (a_n c_n + b_n d_n)$ is then replaced by C_1-summability. The first theorem contains a new proof of an old result of W. H. Young (1910): $f \in V^*$ if and only if the termwise integrated Fourier series of f converges boundedly. Similarly, $f \in (C_1\text{-}V)^*$ if and only if the C_1-sums of the integrated Fourier series converge boundedly. Further characterizations of the same kind concern the space $L_\infty{}^*$ and the Orlicz spaces $(L_\Phi{}^\Delta)^*$ and $(L_\Phi{}^\nabla)^*$, where Δ [or ∇] denotes that the Young function $\Phi(x)$ [or the complementary Young function $\Psi(x)$] satisfies a Δ_2-condition.

The paper also contains theorems on multiplicator sequences: the sequence λ_n is said to belong to (E, E_1) whenever $(\lambda_n a_n, \lambda_n b_n) \in E_1$ for all $(a_n, b_n) \in E$. One of the main results is that for $E = L_p$ $(1 \leq p \leq \infty)$, $L_\Phi{}^\Delta$, C, A, we have $\lambda_n \in (E, C_N)$ if and only if $\Sigma \lambda_n \cos nx \in E^*$. For the same spaces, $\lambda_n \in (E, C)$ if and only if $\Sigma \lambda_n \cos nx \in (C_1\text{-}E)^*$. *A. C. Zaanen* (Leiden)

7393:

Bari, N. K. On subsequences of partial sums of a trigonometric series which converge to zero. Dokl. Akad. Nauk SSSR **129** (1959), 482–483. (Russian)

The author starts from Kozlov's theorem [Mat. Sb. N.S. **26** (68) (1950), 351–364; MR **12**, 174] that there is a nontrivial sine series for which there is a sequence $\{s_{n_k}\}$ of partial sums converging to zero on $[0, \pi]$, uniformly on each $(\delta, \pi - \delta)$, and asks how fast n_k must increase for this to be possible. She conjectures that a necessary condition is that $n_{k+1}/n_k \to \infty$, and states that a function of the kind in question does exist with $n_{k+1}/n_k \leq g(k)$ provided only that $g(k) \uparrow \infty$. The proof depends on two lemmas, of which the second has independent interest. (1) If $0 < \delta < \pi$ and $T_n(x)$ is any sine polynomial of order n, then for any integer p there is an $H(x)$ satisfying: H is odd, $H(x) = 0$ on $[\delta, \pi]$; H and T_n have the same first n Fourier coefficients; H has p continuous derivatives; and if $p = 2n$,

$$\|H(x)\| \leq K^n(\pi/\delta)^{2n}\|T_n(x)\|,$$

where K is an absolute constant. (2) If also $\varepsilon > 0$ there is an absolute constant B such that with $\eta = \delta^2/(\pi^2 B)$ and $\{n/(\nu\eta)\}^{2n} < \varepsilon/\|T_n(x)\|$ (determining ν), there is a sine polynomial Q of order $2\nu - 1$ for which Q has the same first n Fourier coefficients as T_n; $|Q(x)| < \varepsilon$ on $(\delta, \pi]$, and $\|Q(x)\| \leq \eta^{-n}\|T_n(x)\|$.

R. P. Boas, Jr. (Evanston, Ill.)

7394:

Izumi, Shin-ichi. Two theorems concerning convergence of Fourier series. Proc. Amer. Math. Soc. **10** (1959), 744–754.

Let f be an integrable function of period 2π, and let $\varphi_x(t) = f(x+t) + f(x-t) - 2s$. The hypothesis that $\int_0^h \varphi_x(t)dt = o(h)$ as $h \to 0$ in the Lebesgue criterion for the convergence of the Fourier series for f at x to s, is replaced by $\int_0^h \{\varphi_x(t) - \varphi_x(t+h)\}dt = o(h)$ as $h \to 0$. The conclusion is again that of convergence at x, but not necessarily to s.

An additional result is established which shows that a convergence criterion of Noble [Quart. J. Math. Oxford. Ser. (2) **9** (1958), 28–39; MR **21** #1483] is in a sense best possible. *P. Civin* (Eugene, Ore.)

7395:

Weiss, Mary. On a problem of Littlewood. J. London Math. Soc. **34** (1959), 217–221.

L'auteur démontre le théorème suivant: si $n_{k+1}/n_k \geq 3$ $(k \geq 1)$, $\sum a_k{}^2 = \infty$, $a_k(|a_1| + |a_2| + \cdots + |a_k|) = O(1)$, alors

$$1 + \sum \gamma_\nu \cos \nu x = \prod (1 + a_j \cos n_j x)$$

est la série de Fourier-Stieltjes d'une fonction singulière continue, et ses sommes partielles s_λ satisfont à la condition (A) $\int_0^{2\pi} |s_\lambda(x)|dx = O(1)$. On sait que si (A) est satisfait, les coefficients de la série trigonométrique (dont les s_λ sont les sommes partielles) tendent vers zéro [Helson, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 205–206; MR **15**, 788]; cette série est-elle une série de Fourier (problème posé oralement par Littlewood)? Le théorème précédent donne une réponse négative à cette question.

S. Mandelbrojt (Paris)

7396:

Weiss, Mary. The law of the iterated logarithm for lacunary trigonometric series. Trans. Amer. Math. Soc. **91** (1959), 444–469.

L'auteur démontre le théorème suivant: soit

$$S(x) = \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx)$$

une série trigonométrique lacunaire $(n_{k+1}/n_k > q > 1)$, telle que $B_N = (\frac{1}{2}\sum_{k=1}^N (a_k{}^2 + b_k{}^2))^{1/2} \to \infty$ et $a_N{}^2 + b_N{}^2 = o(B_N(\log\log B_N)^{-1/2})$ quand $N \to \infty$; alors, pour presque tout x,

$$V(x) = \limsup_{n\to\infty} (S_N(x)(2B_N{}^2 \log\log B_N)^{-1/2}) = 1.$$

Salem et Zygmund [Acta Math. **91** (1954), 245–301; MR **16**, 467] avaient démontré l'inégalité $(Vx) \leq 1$ p.p. en se restreignant à des n_k entiers; l'auteur reprend et raffine considérablement leurs méthodes. Un théorème analogue concerne les sommes d'Abel de $S(x)$ au lieu des sommes partielles. Une application de ces résultats est donnée dans le papier analysé ci-dessous.

J. P. Kahane (Montpellier)

7397:

Weiss, Mary. On Hardy-Littlewood series. Trans. Amer. Math. Soc. **91** (1959), 470–479.

L'auteur considère les sommes partielles $S_N(\theta)$ resp. $S_N^*(\theta)$ et les sommes d'Abel $S_\rho(\theta)$ resp. $S_\rho^*(\theta)$ des séries $\sum_1^\infty a_n \exp(i\beta n \log n + in\theta)$ avec $a_n = n^{-1/2}$ resp. $a_n = n^{-1/2}(\log n)^{-1/2}$. Elle en étudie avec précision la distribution quand $N \to \infty$ ou $\rho \to 1$; les résultats concernant $S_N(\theta)$ sont les suivants:

$$(1) \qquad \limsup_{N\to\infty} S_N(\theta)/(\log N \log\log\log N)^{1/2} = 1$$

pour presque tout θ; (2) θ représentant une variable aléatoire équipartie sur un ensemble mesurable donné, la distribution bidimensionnelle de $S_N(\theta)/(\frac{1}{2}\log N)^{1/2}$ tend vers la distribution de Gauss.

J. P. Kahane (Montpellier)

7398:

Ganzburg, I. M. The extension of Timan's asymptotic formula to classes of functions with a given modulus of continuity. Dokl. Akad. Nauk SSSR **128** (1959), 447–449. (Russian)

Let f be continuous and let s_n, $\tilde{s}_n$ be its Fourier partial sums and interpolation polynomials of the same order. Let u_n be the s_n transformed by the triangular matrix $(\lambda_k^{(n)})$ in the usual way and $\tilde{u}_n$ the corresponding transforms of $\tilde{s}_n$. For a given subclass of continuous functions consider $\mathcal{E}_n = \sup |f(x) - u_n(x)|$ and $\tilde{\mathcal{E}}_n(x) = \sup |f(x) - \tilde{u}_n(x)|$, sup taken over the subclass. The author determines the asymptotic behavior of $\tilde{\mathcal{E}}_n(x)$ for classes H_ω of functions with moduli of continuity $\omega(t)$. First, if $\omega(t) = Mt^\alpha$ $(0 < \alpha < 1)$, if $\Delta^2\lambda_k^{(n)} \leq 0$ and $|\Delta\lambda_0^{(n)}| = O(1/n)$, then

$$\tilde{\mathcal{E}}_n(x) = (\pi/2)^{\alpha+1}|\sin(n+\tfrac{1}{2})x|\,\mathcal{E}_n \Big/ \int_0^{\pi/2} t^\alpha \sin t\,dt + O(n^{-\alpha}),$$

where the O term is uniform in x. Next, if ω is convex and $(1-\lambda_k^{(n)})/k$ decreases and is concave, then

$$\tilde{\mathcal{E}}_n(x) = \frac{\frac{1}{2}\pi\omega\{2\pi/(2n+1)\}|\sin(n+\frac{1}{2})x|}{\int_0^{\pi/2} \omega\{4t/(2n+1)\}\sin t\,dt}\,\mathcal{E}_n + O(\omega(1/n)).$$

Finally, for arbitrary ω,

$$\tilde{\mathcal{E}}_n(x) = \pi^{-1}\omega(2\pi/(2n+1))|\sin(n+\tfrac{1}{2})x| \sum_{k=0}^{n} \frac{\lambda_k^{(n)}}{n-k+1} + O(\omega(1/n)),$$

whence there follows an asymptotic formula for $\mathcal{E}_n$ when ω is convex.

R. P. Boas, Jr. (Evanston, Ill.)

7399:

Hartman, Philip. On completely continuous Hankel matrices. Proc. Amer. Math. Soc. **9** (1958), 862–866.

If $x = (x_0, x_1, \cdots)$ is a sequence of complex numbers, let $|x|$ denote the norm $(\sum |x_n|^2)^{1/2}$. For a sequence $f = (f_0, f_1, \cdots)$ of finite norm, define the Hankel matrix $H = H(f) = (f_{n+m})$ and the bilinear form $H(x, y) = \sum\sum f_{n+m}x_n y_m$. If H is bounded in the sense of Hilbert, set $\|H\| = \sup |H(x, y)|$ for $|x| = |y| = 1$.

Toeplitz had shown that $H(f)$ is bounded if either of the functions $F_1(t) \sim f_0/2 + \sum f_n \cos nt$ or $F_2(t) \sim \sum f_{n-1} \sin nt$ is bounded on $(0, 2\pi)$. In a recent paper [Ann. of Math. (2) **65** (1957), 153–162; MR **18**, 633] the reviewer extended this result to the following necessary and sufficient condition: $H(f)$ is bounded if, and only if, there exists a $g = (0, g_1, \cdots)$ of finite norm such that

$$(*) \qquad \sum f_n e^{-int} + \sum g_n e^{int}$$

is the formal Fourier series of a function $F(t)$ bounded on $(0, 2\pi)$; moreover, there exists a particular g such that $\|H\| = \sup |F(t)|$.

In an earlier paper with Wintner [Amer. J. Math. **72** (1950), 359–366; MR **12**, 187], the author had shown that $H(f)$ will be completely continuous if either $F_1(t)$ or $F_2(t)$ is a continuous function. He now proves that a necessary and sufficient condition for $H(f)$ to be completely continuous is the existence of a g such that (*) is the formal Fourier series of a continuous function.

Z. Nehari (Pittsburgh, Pa.)

7400:

Hewitt, Edwin; and Zuckerman, H. S. Some theorems on lacunary Fourier series, with extensions to compact groups. Trans. Amer. Math. Soc. **93** (1959), 1–19.

Denote by $f^\wedge$ the Fourier transform (i.e., sequence of Fourier coefficients) of a function or measure f on the line modulo 2π, and by N a set of integers. It is known [see, e.g., Zygmund, *Trigonometrical Series*, Subwencjl Funduszu Kultury Narodowej, Warszawa-Lwów, 1935; Chapters VI and IX] that if N admits lacunary gaps, functions or measures f which are N-spectral in the sense that $f^\wedge$ vanishes outside N enjoy several extraordinary properties. One type of property is exemplified by these: any N-spectral bounded function has an absolutely convergent Fourier series; any N-spectral integrable function, or even any N-spectral measure, is a square-integrable function. Another type of property is of the following sort: every bounded sequence on N [resp. every sequence on N which tends to zero at infinity] is the restriction to N of $f^\wedge$ for some measure [resp. integrable function] f.

The paper under review sets out to show that, without any direct assumption about the lacunary character of N, properties of the above type can be grouped into equivalent sets, the proofs being almost purely functional analytic in character. Only very simple properties of Fourier series are called upon. In addition, some novel conditions of an arithmetical nature are given as sufficient to ensure that N has the extraordinary properties considered. Advantage is taken of the abstract type of proof to make a relatively effortless extension to general compact groups.

{Reviewer's notes. The functional analytic content of the arguments employed may be summarized thus: Let E, F be Banach spaces, u a continuous linear map of E into F, E', F' the duals of E and F, u' the adjoint of u; take strongly closed vector subspaces L and M of E' and suppose that L contains $u'(F')$ and is contained in the union of the weak closures of bounded parts of $u'(F')$. Then (1) $u(E)$ is closed in F if and only if $u'(F') = L$; (2) $u(E) = F$ if and only if $u'(F') = L$ and u' is 1-1; (3)

putting $H=u'^{-1}(M)$, $v=u'|H$ (qua continuous linear map of H into M), then $u(E)=F$ implies that for each y' in F' one has $y'|H=v'(z')$ for some z' in M', and the converse is true provided H is strongly dense in F'. One may take for example $E=L^1$, $F=c_0(N)$, $E'=L^\infty$, $F'=l^1(N)$, u defined by the Fourier transformation restricted to N, $M=C$ (continuous functions), and L the N-spectral bounded [or continuous] functions. There results then Theorem 2.1 of the paper under review; Theorem 6.1 also appears by similar specialisation.}

R. E. Edwards (Reading, Berks)

7401:

König, Heinz. Bemerkungen zur Fourier-Stieltjes-Transformation. Arch. Math. **10** (1959), 137–143.

Let $\varphi=\varphi(u)$ be a real-valued function of bounded variation defined for real u. It is assumed that $\varphi(u)$ is between $\varphi(u-0)$ and $\varphi(u+0)$. If φ is absolutely continuous then, by classical results of Riemann and Lebesgue, the Fourier-Stieltjes transform $P(t)=\int_{-\infty}^{\infty} e^{iut}d\varphi(u)$ satisfies the limit relation (1) $\lim_{t\to\infty} P(t)=0$.

The object of the paper is to establish less restrictive conditions which are sufficient for the validity of (1). It is shown (Satz 3) that the following condition is sufficient: There is a function ψ of bounded variation which has the following two properties:

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} P(t-u)d\psi(u) = 0; \tag{2}$$

(3) if $N=N(\psi)$ denotes the set of real zeros of the Fourier-Stieltjes transform of ψ, then $\mu(N)=0$ where μ is the Borel measure generated by the variation of φ. In particular, these conditions on ψ are satisfied if (2) is true, and N consists only of isolated points at which $\varphi(u)$ is continuous.

For the proof of Satz 3 the author proves the following more general result (Satz 4): Let ν denote the signed measure generated by φ. Let ψ be of bounded variation and satisfy (2). Let N and μ be defined as above. Then

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} e^{iut}f(u)d\nu(u) = 0$$

for every Borel measurable and bounded function f which is zero on N, with the possible exception of a subset of N of μ-measure 0.

Finally the preceeding results are used to give a sufficient condition for certain dissipative linear transformations to be "continuous at infinity" [see the review #7410 below]. *E. H. Rothe* (Ann Arbor, Mich.)

7402:

Cinquini, Silvio. Sopra qualche concetto della teoria delle funzioni quasi-periodiche. Matematiche. Catania **12** (1957), 1–17.

Conférence, avec de nombreuses références bibliographiques, sur les fonctions presque-périodiques: définition originale de Bohr, séries de Fourier, unicité, approximation par le procédé de Fejér-Bochner, généralisations de Stepanoff et Kovanko, fonctions normales, applications aux équations différentielles. *J. Favard* (Paris)

7403:

Günzler, Hans. Fastperiodische Lösungen linearer hyperbolischer Differentialgleichungen. Math. Z. **71** (1959), 223–250.

An analytical most periodic function $F(z)=F(x+iy)=F(x, y)$ in a strip $c<x<d$, $-\infty<y<\infty$, satisfies the potential equation $\Delta F=0$, and is uniformly for $c<x<d$ almost periodic in y. In the present paper the potential equation is replaced by a hyperbolic differential equation

$$F_{xx}(x, y)-F_{yy}(x, y)=a(x)F(x, y), \tag{1}$$

where $a(x)$ is finite in $c<x<d$ and integrable in compact subsets. Generalized solutions of (1) are defined by passing from (1) to a certain integral equation. Thus differentiability assumptions are avoided; further, using a norm $\|F\|_\rho=\sup_{x,y}|\rho(x)F(x, y)|$, the generalized solutions of (1) with $\|F\|_\rho<\infty$ form a Banach space $\mathfrak{F}_\rho$. The "normfunction" $\rho(x)$ has only to be chosen with a positive lower bound in each compact interval, but there exists to every differential equation (1) a universal normfunction $\rho_0(x)$ with the property that if $\|F\|_\rho<\infty$ for some ρ, then $\|F\|_{\rho_0}<\infty$. An almost periodic vector from $\mathfrak{F}_\rho$ is defined as a generalized solution of (1), which "ρ-uniformly in x" is almost periodic in y. More precisely: To every $\varepsilon>0$ there shall exist a finite covering of the set of real numbers $R=\bigcup_{\nu=1}^{n} A_\nu$, such that $\|F(x, y+c)-F(x, y+d)\|_\rho<\varepsilon$, when only c and d belong to the same A_ν. As a main result, it is proved that every almost periodic generalized solution F of (1) with $\|F\|_\rho<\infty$ can be ρ-norm-approximated by finite linear combinations of elementary functions (eigenfunctions) of the form $h_\omega(x)e^{i\omega y}$, where the functions $h_\omega(x)$ satisfy the ordinary differential equation $h''(x)-(a(x)-\omega^2)h(x)=0$ belonging to (1). A decisive tool in the proof is Maak's approximation theorem for vector-valued almost periodic functions [Math. Ann. **122** (1950), 157–166; MR **12**, 508].

Almost periodicity is also characterized by translation numbers; Fourier series of the functions are introduced and studied.

The last section contains various examples of differential equations of the type (1), and the corresponding almost periodic vectors. One of the examples comprises (after a simple transformation) the type of differential equation considered by Delsarte for a similar purpose of generalizing almost periodic functions [Acta Math. **69** (1938), 259–317]. *E. Følner* (Copenhagen)

7404:

Rudin, Walter. Measure algebras on abelian groups. Bull. Amer. Math. Soc. **65** (1959), 227–247.

Dans cette conférence, faite à la Mathematical Society, l'auteur décrit quelques développements récents de l'analyse générale de Fourier, en appuyant surtout sur son aspect algébrique. Il note que, souvent, des problèmes qui se posent en termes algébriques sont transformés en problèmes pouvant être résolus par des méthodes de l'analyse classique, la solution reprenant l'aspect algébrique. L'auteur se limite aux questions auxquelles il a contribué lui-même, d'une manière fort importante d'ailleurs. *S. Mandelbrojt* (Paris)

7405:

Glicksberg, Irving. Convolution semigroups of measures. Pacific J. Math. **9** (1959), 51–67.

Let S be a compact topological semi-group, C the Banach space of all complex-valued continuous functions on S, M the Banach space of all Radon measures on S, M_1

the unit ball of M and $\tilde{M}$ the set of all positive measures belonging to M of total mass one. The sets M_1 and $\tilde{M}$ are compact semi-groups when endowed with the usual convolution and the topologies induced by $\sigma(M, C)$. The main purpose of this paper is the determination of all closed subgroups of M_1 and $\tilde{M}$ in the case when the semi-group S is abelian. The (closed) subgroups of the convolution semi-group consisting of all bounded Radon measures of norm inferior to one on a locally compact abelian group are also determined.

C. Ionescu Tulcea (New Haven, Conn.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 7369, 7383, 7447.

7406:

Benson, Donald C. Extensions of a theorem of Loewner on integral operators. Pacific J. Math. **9** (1959), 365–377.

Let $x=x(t)$, $y=y(t)$ be continuous for $-\infty<t<\infty$. The curve $x=x(t)$, $y=y(t)$ is said to be of weakly positive sense if for every P not in the closure of the curve there exists a $K=K(P)\leqq 0$ such that $\theta_P(\delta)-\theta_P(\gamma)\geqq K$ for all γ, δ $(\gamma\leqq\delta)$. Here $\theta_P(t)$ denotes a continuous argument $(-\infty<t<\infty)$ of the vector from P to $(x(t), y(t))$. The following extension of a theorem of Loewner [Ann. of Math. (2) **49** (1948), 316–332; MR **9**, 502] is proved.

Let $k(t)$ be integrable on the interval $[0, \infty)$. Consider the curves $(x(t), y(t))$ generated by applying the operator $y(t)=-\int_0^\infty k(\tau)x(t-\tau)d\tau$ to all almost periodic functions $x(t)$. The functions $y(t)$ are also almost periodic. A necessary and sufficient condition that the curves generated in this way are all of weakly positive sense is that $k(t)$, after a possible change on a set of measure zero, is analytic in $[0, \infty)$, and that $k'(t)$ has a representation as a Laplace-Stieltjes integral $k'(t)=\int_0^\infty e^{-st}d\mu(s)$ with non-decreasing determining function $\mu(s)$.

Other extensions of Loewner's theorem are indicated. Their proofs are given elsewhere.

E. Følner (Copenhagen)

7407:

Džrbašyan, M. M. Quasi-isometric mapping of the spaces of functions $L_{\sigma_1}^2(a_1, b_1)$, $L_{\sigma_1}^2(a_2, b_2)$. Dokl. Akad. Nauk SSSR **128** (1959), 456–459. (Russian)

A theorem of Bochner and Chandrasekharan [*Fourier transforms*, Princeton Univ. Press, Princeton, N.J., 1949; MR **11**, 173; pp. 150–156], characterising integral kernels which induce unitary transformations of $L^2(a, b)$ into itself, is generalised to characterise kernels giving isometric mappings of $L_{\sigma_1}^2(a_1, b_1)$ onto $L_{\sigma_2}^2(a_2, b_2)$. Kernels which approximate a kernel of this type in a certain sense correspond in a one-one fashion to quasi-isometric transformations—i.e., the ones in which the ratio of the norm of the transform to that of the original vectors differs from 1 by less than a fixed amount.

J. L. B. Cooper (Cardiff)

7408:

Borodachev, N. M. A generalized formula for the transform of a product of functions. J. Appl. Math. Mech. **22** (1958), 997–998 (706 Prikl. Mat. Meh.).

The author gives a generalisation of a result of Natanzon's [Prikl. Mat. Meh. **20** (1956), 671–672] for the Laplace transform of a product of functions to the case when one of the functions in the product is a function of a function. Transforms for the product and quotient of two functions are obtained as special cases.

W. D. Collins (Newcastle-upon-Tyne)

7409:

Mikusiński, J. On the function whose Laplace-transform is e^{-s^α}. Studia Math. **18** (1959), 191–198.

For the function $F_\alpha(t)$ of the title, with $0<\alpha<1$, an integral representation is derived from the complex inversion formula for the Laplace-transform by a change of contour that makes the integrand real and positive. This enables the author to give asymptotic formulas for $F_\alpha(t)$, both for $t\to 0+$ and for $t\to\infty$. {Unknown to the author, F. M. Ragab [Comm. Pure Appl. Math. **11** (1958), 115–127; MR **20** #7189] had obtained the asymptotic behaviour of $F_\alpha(t)$ for $t\to 0+$ by a complicated use of MacRobert's E function, but only in the case $\alpha=m^{-1}$, where $m=2, 3, 4, \cdots$.} *L. A. Rubel* (Urbana, Ill.)

7410:

König, Heinz; und Meixner, Josef. Lineare Systeme und lineare Transformationen. Math. Nachr. **19** (1958), 265–322.

The introduction contains remarks concerning the physical motivation as well as the history of the subject treated.

In section 1 some of the basic definitions are given: For $m=0, 1, 2, \cdots, \infty$, let C_m denote the linear space of all functions f of the real variable t which have the following two properties: (i) f is m times continuously differentiable for all t; (ii) there exists a number $\alpha=\alpha(f)$ such that $f(t)=0$ for $t\leqq\alpha$. A map $L: C_m\to C_0$ is called a linear transformation defined on C_m if the following three conditions are satisfied. (I) $L(a_1f_1+a_2f_2)=a_1Lf_1+a_2Lf_2$ for any couple f_1, f_2 in C_m and any couple of real numbers a_1, a_2. (II) If $f\in C_m$ and $g(t)=f(t-\vartheta)$ for all t and fixed ϑ, then $(Lg)(t)=(Lf)(t-\vartheta)$. (III) If the element $f\in C_m$ has the property $f(t)=0$ for $t\leqq\alpha$, then $(Lf)(t)=0$ for $t\leqq\alpha$.

Various equivalent definitions of the continuity of order p $(p=0, 1, \cdots, m)$ of a linear transformation L defined on C_m are given. We quote one called B_p: For p finite, B_p requires: to every positive τ there corresponds a non-negative M such that (1) $|(Lf)(t)|\leqq M$ for $t\leqq\tau$ for all $f\in C_m$ which vanish for $t\leqq 0$ and for which $|f^{(p)}(t)|\leqq 1$ for $t\leqq\tau$. B_∞ requires: to each positive τ there corresponds a non-negative integer q and a non-negative M such that (1) holds for all $f\in C_\infty$ which vanish for $t\leqq 0$ and for which $|f^{(q)}(t)|\leqq 1$ for $t\leqq\tau$. For finite p the inf of all M satisfying (1) is called the norm of order p of L and denoted by $\|L\|_p(\tau)$.

Section 2 deals with representations of transformations L which are continuous of order m. These representations are of the form (2) $(Lf)(t)=\int_0^\infty f^{(m)}(t-u)d\Phi(u)$ where $\Phi(u)$ in every finite interval of the non-negative part of the u-axis is of bounded variation and satisfies certain continuity conditions.

In section 3, "slowly increasing" functions and linear transformations are discussed. A function $h(t)$ defined for $\alpha\leqq t<\infty$ is called slowly increasing if, for $t\geqq\alpha$, $|h(t)|$ is majorized by some polynomial. A linear transformation L

defined on C_∞ is called slowly increasing if $Lf(t)$ is a slowly increasing function for every $f \in C_\infty$ which is zero outside some finite interval. Of the theorems proved in this section we mention the following one: A linear transformation L defined on C_∞ and continuous of order ∞ is slowly increasing if and only if the following condition is satisfied: There exists a non-negative integer m such that L is continuous of order m while $\|L\|_m(\tau)$ as function of τ is slowly increasing for positive τ. The smallest such m is called the index $m(L)$ of L.

Section 4 deals with the continuation of a linear transformation L defined on C_m $(m=0, 1, 2, \cdots, \infty)$ and continuous of order m to a linear transformation defined on a space $\bar{C}_m$ which is defined as the linear space of all real-valued functions $f(t)$ which, together with the derivatives of order 1, 2, $\cdots$, m, is continuous and "quickly decreasing". Here a function $h(t)$ is called quickly decreasing (for $t \to -\infty$) if $\lim_{t\to-\infty} P(t)h(t)=0$ for every polynomial $P(t)$. Of the results of this section we quote the following one which, with a suitable definition of the continuity of a linear transformation defined on $\bar{C}_m$, states a necessary and sufficient condition for the possibility of extending a linear transformation L defined on C_m and continuous of order m to a linear transformation defined on $\bar{C}_m$ and continuous of order m: If $m=\infty$ the condition is that L is slowly increasing; if m is finite the condition is that $\|L\|_m(\tau)$ for $\tau>0$ is a slowly increasing function of τ. In this latter case the representation (2) still holds if $f \in \bar{C}_m$ and if L is replaced by its extension to $\bar{C}_m$.

In section 5 with each slowly increasing transformation L defined on C_∞ and continuous of order ∞ is associated an "impedance" function $Z(s)$ in the following way. Let $m(L)$ be the index of L defined above, let m be an integer $\geq m(L)$, and let Φ be the function occurring in the representation (2) of L. Then

$$Z(s) = s^{m-1}\int_0^\infty e^{-st}d\Phi(t) \quad (\Re(s) > 0),$$

where $\Re(s)$ denotes the real part of s. $Z(s)$ turns out to be independent of m. If $\Lambda(\cdot)$ denotes the Laplace transform operator then the relation

$$(\Lambda Lf)(s) = sZ(s)(\Lambda f)(s) \quad (\Re(s) > 0)$$

is proved under the following assumptions: L is a linear transformation defined on $\bar{C}_m$ (m a finite non-negative integer) and continuous of order m; $\|L\|_m(\tau)$ is slowly increasing for $\tau>0$; finally $f \in C_m$, $f(t)=0$ for $t \leq 0$, and $f^{(m)}(t)$ is slowly increasing.

In preparation for the later parts of the paper section 6 deals with "positive" functions $Z(s)$ of the complex variable s, i.e., with functions Z having the following properties: $Z(s)$ is defined and analytic for $\Re(s)>0$, $\Re(\bar{Z}(s))>0$ for $\Re(s)>0$, $Z(s)$ is real for s positive. Of the results of section 6 we mention the following one: $Z(s)$ is positive if and only if it admits a representation of the form

$$(3) \quad Z(s) = As+\frac{B}{s}+\int_{-\infty}^{\infty}\frac{1-i\rho s}{s-i\rho}\,d\varphi(\rho) \quad (\Re(s) > 0),$$

where A, B are non-negative constants, and φ is a distribution function (in the sense of probability theory) with certain prescribed properties.

Section 7 deals with positive (or dissipative) transformations on C_∞, i.e., transformations L which, in addition to the conditions (I), (II), (III) (with $m=\infty$) of section 1, satisfy

$$(IV) \qquad \int_{-\infty}^{\tau} Lf(t)f'(t)dt \geq 0 \quad \text{for all } \tau.$$

It is assumed that L is not the zero transformation. The close connection between the positivity of a linear transformation L and the positivity of functions $Z(s)$ is expressed in the following statement: If L is a linear positive slowly increasing transformation defined on C_∞ and continuous of order ∞, then the impedance function $Z(s)$ of L is positive (Satz 18). Conversely, to a given positive $Z(s)$ there exists one and only one L of the properties just mentioned such that Z is the impedance function of L (Satz 19).

Other questions treated in this section are those of the representation, continuation and the inversion of slowly increasing positive transformations L. The method of dealing with the inversion problem may be sketched as follows. If $Z(s)$ is the impedance function of L, then the function $Y(s)$ defined by $Y(s)Z(s)=1$ for $\Re(s)>0$ is positive. The transformation L^0 (defined in a proper space) which has $Y(s)$ as impedance function is proved to have the property $L^0Lf(t)=LL^0f(t)=f''(t)$, and the inverse L^{-1} of L is then obtained by integrating twice.

Section 8 deals with linear positive transformations defined on C_∞ which are "continuous at infinity". This latter condition is more restrictive than the condition of being slowly increasing, used in section 7; it is defined as follows: $\lim Lf(t)$ exists (as a finite value) for every $f \in C_\infty$ which is constant from a certain t on. Of the results obtained in this section we quote the following. Let L be a positive linear slowly increasing transformation defined on C_∞ which is continuous of order ∞. Let φ be the distribution-function appearing in the representation (3) of the positive impedance function $Z(s)$ associated with the positive L. Then the necessary and sufficient condition for L to be continuous at infinity is that (4) $\lim_{t\to\infty} P(t)=0$, where $P(t)$ is the characteristic function (Fourier-Stieltjes transform) of φ. Necessary for the validity of (4) is that $\varphi=\varphi(t)$ is continuous for all t while the condition that φ is absolutely continuous on every closed bounded interval is sufficient.

In conclusion, applications of the preceding theory to various physical systems are discussed.

E. H. Rothe (Ann Arbor, Mich.)

7411:

★**Churchill, Ruel V. Operational mathematics.** 2nd ed. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. ix+337 pp. $7.00.

This is a revised edition of *Modern operational mathematics in engineering* [McGraw-Hill, New York, 1944; MR 5, 267]. Appreciable modification of the basic theory has been made to provide a more general and integrated approach. A foretaste of this arises at the outset by the introduction of the Laplace transform through the concept of linear operators. The main major changes are the chapter on Sturm-Liouville systems which goes much deeper than that in the first edition, and the chapter on Fourier transforms, which not only deals with particular transforms on finite and unbounded regions, but also discusses the selection of the appropriate transform for a general second order linear differential operator. Other

changes comprise new applications such as servomechanisms, the derivation of the partial differential equations representing physical situations, and many new examples.

Like its predecessor, this book provides an excellent introduction to the theory of the Laplace transform and other integral transforms, and their application to ordinary and partial differential equations and integral equations arising in physical problems. In common with other treatments of the application of transforms, the reviewer feels that in certain instances where this elegant but more elaborate method yields solutions deducible by simpler methods, for example Fourier series or the general solution of the wave equation, it would be useful to point this out. In the case of the wave equation, there are examples in which the existence of discontinuities in derivatives leads to difficulties in interpretation of the wave equation itself, quite apart from the interpretation of the associated transforms, which could be clarified. However, these detailed comments concern fine points, which do not detract perceptibly from an outstanding presentation. *E. H. Lee* (Providence, R.I.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 7433.

7412:

Ghermanesco, Michel. Équations intégrales aux deux limites variables. C. R. Acad. Sci. Paris **249** (1959), 1606–1607.

Continuing his preceding note [same C. R. **248** (1959), 1104–1105; MR **21** #796], the author considers

$$\varphi(x) = \lambda \int_{-x}^{x} K(x, s)\varphi(s)ds + f(x),$$

and reduces its solution to that of a pair of integral equations. "En ce qui concerne l'existence et la multiplicité des solutions, les résultats sont tellement différents, si l'on passe d'un exemple à l'autre, qu'il est difficile d'en tirer une conclusion générale." He illustrates this remark with the equations

$$\text{(a)} \qquad \varphi(x) = \lambda \int_{-x}^{x} K(s)\varphi(s)ds + f(x),$$

$$\text{(b)} \qquad \varphi(x) = \lambda \int_{-x}^{x} (\cos s + \cot s)\varphi(s)ds + \sin^m x.$$

With $K(x)=x^3$, $f(x)=x^4$, (a) has the solution $\varphi=f$ for every λ and the corresponding homogeneous equation does not always have a nontrivial solution. For (b), if $2\lambda-m-1$ and $m-2\lambda$ are not zero, there is a unique solution when λ is not half an odd integer and there are an infinity of solutions when λ is half an odd integer; the corresponding homogeneous equation has a nontrivial solution when λ is half an odd integer.

R. P. Boas, Jr. (Evanston, Ill.)

7413:

Neĭmark, Yu. I. On the permissibility of linearisation in studying stability. Dokl. Akad. Nauk SSSR **127** (1959), 961–964. (Russian)

Let $u(t)$, $\omega(t)$ be functions with values in normed linear spaces U, Ω, and let $G(t, \tau)$ be a linear operator such that

$$(*) \qquad u(t) = \int_{-\infty}^{t} G(t, \tau)[f(\tau, u(\tau)) + \omega(\tau)]d\tau.$$

The author calls (*) a dynamical system. Theorem I: If $f \equiv 0$ and

$$\sup_{t_1 \geq t_0} \int_{t_0}^{t_1} \|G(t_1, \tau)\|d\tau < +\infty,$$

then given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|u(t)\| < \varepsilon$ for all t whenever $\|\omega(t)\| < \delta$ for all t. Theorem II: If $f \not\equiv 0$ and if a solution is known for $t \leq t_0$, then it has a unique continuous continuation to $I = [t_0, t_1]$ whenever

(i) $f(\tau, 0) = 0$, $\|f(\tau, x) - f(\tau, y)\| < A\|x - y\|$ $(\tau \in I;\ \|x\|, \|y\| < M)$;

(ii) $\left\| \int_{-\infty}^{t} G(t, \tau)[f(\tau, u(\tau)) + \omega(\tau)]d\tau \right\| < M' < M$ on I;

(iii) $\sup_{t \in I} \int_{t_0}^{t_1} \|G(t, \tau)\|d\tau = G_{t_0 t_1}$, $AG_{t_0 t_1} < 1$;

(iv) $\|\omega(t)\| < G_{t_0 t_1}^{-1}[M(1 - AG_{t_0 t_1}) - M']$ on I.

Moreover, this continuation can be obtained by successive approximations, and

$$\|u(t)\| < [M' + G_{t_0 t_1} \sup_{t \in I} \|\omega(t)\|](1 - AG_{t_0 t_1})^{-1}.$$

Theorem III: If for $f \equiv 0$ the assertion of (I) holds then it also holds whenever f satisfies (i) and $AG_{-\infty,+\infty} < 1$. Moreover, if $\|\omega(t)\| < MG_{-\infty,+\infty}^{-1}(1 - AG_{-\infty,+\infty})$ then

$$\|u(t)\| < G_{-\infty,+\infty}(1 - AG_{-\infty,+\infty})^{-1} \sup_{\tau \leq t} \|\omega(\tau)\|.$$

H. A. Antosiewicz (Los Angeles, Calif.)

7414:

Vaĭnberg, M. M.; and Šragin, I. V. Nonlinear operators and Hammerstein's equations in Orlicz spaces. Dokl. Akad. Nauk SSSR **128** (1959), 9–12. (Russian)

This paper states theorems on the existence and uniqueness of solutions of the non-linear equation

$$(1) \qquad u(x) = \int_B K(x, y)g(u(y), y)dy \ (\equiv \Gamma u(x))$$

in certain Orlicz spaces without assuming that the linear integral operator A generated by the kernel K is completely continuous. Six theorems are stated, but cannot be completely announced here. As a sample result, it is asserted that if A gives rise to a positive self-adjoint operator A_0 on L^2, if $g(u, x)$ is non-decreasing in u for almost all fixed x in B, and if

$$2\int_0^u g(v, x)dv \leq au^2 + b(x)|u|^\alpha + c(x)$$

with $0 \leq a\|A_0\| < 1$, $0 < \alpha < 2$, then equation (1) has a unique solution in a certain Orlicz space. The proof is said to flow from variational considerations.

R. G. Bartle (Urbana, Ill.)

7415:

Hull, T. E. On some integral equations with unbounded eigenvalues. J. Soc. Indust. Appl. Math. **7** (1959), 290–297.

The integral equation with kernel $K(x,t)=x^{-1}e^{-t/x}$, $0\leqq x$, $t<\infty$, is studied, and it is observed that for $n=0, 1, 2, \cdots$ the function x^n is an eigenfunction corresponding to the eigenvalue $n!$. Theorem 1: If $\lambda\neq -1/n!$ and $f(x)=\sum_0^\infty c_n x^n$ is an entire function of x, then (i) $y(x)=\sum_0^\infty c_n x^n/(1+\lambda n!)$ is a solution of $y=f-\lambda Ky$. Unless $f(x)$ is a polynomial, this solution $y(x)=y(x,\lambda)$ has an essential singularity at $\lambda=0$, and hence cannot be represented by the Neumann series, even if the series converges. This convergence can actually occur, as shown by theorem 2: If $|\lambda|<1$, and $f(x)$ is bounded for $x\geqq 0$ and integrable over any bounded interval, then there exists a unique solution, bounded for $x\geqq 0$ and integrable over any bounded interval, and this solution is the sum of the Neumann series. For $f(x)=e^{-x}$ both theorems are applicable; the uniqueness in theorem 2 implies that $y(x)$ in (i) is unbounded; we have $y(x)=e^{-x}+o(\lambda)$ as $\lambda\to 0$, $|\arg\lambda|<\pi$, provided $|x|<1$, and the first term of the Neumann series is also e^{-x}. Two other kernels showing similar features are mentioned, and some remarks are made concerning the solution of $y=f-Ky-K_1y$, where K_1 is small in a certain sense. *A. C. Zaanen* (Leiden)

FUNCTIONAL ANALYSIS

See also 7201, 7382, 7399, 7400, 7405, 7410, 7411.

7416:

Rolewicz, S. Remarks on linear metric Montel spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 195–197. (Russian summary, unbound insert)

Let X be a B_0-space (=linear metric locally convex complete space). Let $\Phi(X)$ be the family of real-valued functions on $(0, \infty)$ that serve to define the "approximate dimension" of X [Kolmogorov, Dokl. Akad. Nauk SSSR **120** (1958), 239–241; MR **20** #4171]. Among the theorems announced in this note are: (1) X is a Schwartz space if and only if $\Phi(X)$ is not empty. (2) If X is an infinite-dimensional Montel space, then it contains an infinite-dimensional nuclear subspace with an unconditional basis. (3) If X is a nuclear space with an unconditional basis and if $\lim_{\varepsilon\to 0} e^{\lambda/\varepsilon}/\phi(\varepsilon)=0$ for some $\lambda>0$, then $\phi\in\Phi(X)$. Proofs appear in Studia Math. **18** (1959), 1–9 [MR **21** #2183]. *M. Jerison* (Lafayette, Ind.)

7417:

Slugin, S. N. A two-sided estimate for the solution of a linear functional equation. Dokl. Akad. Nauk SSSR **127** (1959), 34–36. (Russian)

The problem posed is to obtain two-sided bounds ($a\leqq x\leqq b$) for the solution x of a functional equation in a partially ordered space X, by use of an auxiliary space $\bar{X}$, the point being that $\bar{X}$ may be a simpler space than X, which would facilitate the practical application. Let X be a complete vector lattice (a K-space in the author's notation, which, together with other notation, is taken from the frequently cited book of Kantorovič, Vulih and Pinsker, *Funkcional'nyĭ analiz v poluuporyadočennyh prostranstvah* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **12**, 340]); X' a complete partially ordered subspace such that to each $x\in X$ there correspond a, $b\in X'$ with $a\leqq x\leqq b$ in X; and $\bar{X}$ a K-space isomorphic to X': $\varphi\bar{X}=X'$. Let H, Γ be operators carrying X, X' into themselves, respectively, and determine $\bar{\Gamma}$ taking $\bar{X}$ into itself by the relation $\bar{\Gamma}\varphi^{-1}x'=\varphi^{-1}\Gamma x'$.

For the equation (1) $Hx=y$ there is the following theorem. Let H, Γ be additive and positive-reversible, $H^{-1}>0$, and let $H\geqq\Gamma$ on X'. Choose u, $v\in\bar{X}$ arbitrarily to satisfy $\varphi u\leqq -y_-$, $\varphi v\geqq y_+$, and let $x=x^*$ satisfy (1). Then $\varphi\bar{\Gamma}^{-1}u\leqq x^*\leqq\varphi\bar{\Gamma}^{-1}v$. For equation (2) $x-Hx=y$ there is a corresponding result similar to one in the book cited above, pages 474–475. Application is made to functional equations of the form $x(s)-\lambda SH(s,t)x(t)=y(s)$, where $\lambda>0$; S is any one of the operators $\sum_{i=1}^m$, $\sum_{i=1}^\infty$, $\int_0^1\cdot dt$, $\int_0^\infty\cdot dt$; and $S|H(s,t)|<\infty$.

I. M. Sheffer (University Park, Pa.)

7418:

Mikołajska, Z.; et Ważewski, T. Sur les opérations transformant séries convergentes en séries convergentes. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 615–618.

Let f be a function from X to Y where X and Y are Banach spaces. The authors prove the following two conditions are equivalent: (1) there exists F, a continuous linear transformation, such that $f\equiv F$ in a neighborhood of the origin; (2) if $\sum x_n$ is a convergent series, then $\sum f(x_n)$ is a convergent series.

L. Brown (Detroit, Mich.)

7419:

Makarov, B. M. The problem of moments in some functional spaces. Dokl. Akad. Nauk SSSR **127** (1959), 957–960. (Russian)

Let X be a locally convex space, let $(x_n, n\geqq 0)$ be a sequence of elements from X and let (μ_n) be a sequence of real numbers. The moment problem is the problem of finding a functional f in X^* such that $f(x_n)=\mu_n$, $n\geqq 0$. Let X be the inductive limit of an expanding sequence of B-spaces X_n with closed unit balls denoted by S_n. The sequence (X_n) satisfies condition (F) if for any $\lambda_k>0$ the set $\lambda_0S_0+\lambda_1S_1+\cdots+\lambda_nS_n$ is closed in X_{n+1}, $n\geqq 0$. It is asserted that if (X_n) satisfies condition (F), and the x_n are linearly independent vectors whose span L is a subspace of X such that $L\cap X_n$ is finite dimensional for all $n\geqq 0$, then the moment problem has a solution for all sequences (μ_n). Applications to several concrete spaces are given. As one application, a moment theorem in $C(-\infty,\infty)$ with $x_n(t)=t^n$ which is due to G. Pólya [C. R. Acad. Sci. Paris **207** (1938), 708–711] is obtained.

R. G. Bartle (Urbana, Ill.)

7420:

Agaev, G. N.; and Yagubov, S. Ya. Extension of linear functionals in linear spaces. Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Mat. Teh. Nauk **1959**, no. 3, 29–38. (Russian. Azerbaijani summary)

A. E. Taylor [Duke Math. J. **5** (1939), 538–547; MR **1**, 58] proved that if the unit sphere in the conjugate space of a normed linear space E is strictly convex, then every bounded linear functional on an arbitrary subspace of E has a unique bound-preserving extension to all of E. The converse of this theorem was proved by S. R. Foguel

[Proc. Amer. Math. Soc. **9** (1958), 325; MR **20** #219]. In the paper under review, these results are generalized by replacing the norm on E with a subadditive, positive-homogeneous functional. *M. Jerison* (Lafayette, Ind.)

7421:

Singer, I. Elementary proof of a theorem of S. R. Foguel on biorthogonal systems in Banach spaces. Rev. Math. Pures Appl. **3** (1958), 305–307.

By elementary methods (avoiding the Baire category theorem) the author proves a slightly weaker version of a theorem of S. R. Foguel [Pacific J. Math. **7** (1957), 1065–1072; MR **20** #7211]. *L. Brown* (Detroit, Mich.)

7422:

Okano, Hatsuo. L'intégration des fonctions à valeurs vectorielles d'après la méthode des espaces rangés. Proc. Japan Acad. **35** (1959), 77–82.

Extension de l'intégrale (E. R.) de Kunugi à des fonctions dont les valeurs appartiennent à un espace vectoriel. *A. Appert* (Angers)

7423:

Batt, Jürgen; und König, Heinz. Darstellung linearer Transformationen durch vektorwertige Riemann-Stieltjes-Integrale. Arch. Math. **10** (1959), 273–287.

Let I be the finite closed interval $[\alpha, \beta]$ of the real line. Let E and F be Banach spaces. Let $C(E)$ be the Banach space of all continuous $f: I \to E$, with the usual supremum norm. The Banach space of all bounded linear mappings of E into F is denoted by $L(E, F)$. A function $g: I \to E$ is defined to be of bounded variation in the obvious way, with norms replacing absolute values. The total variation of g is written $\operatorname{Var} g$. The "reduced variation" of g, written $\mathrm{R}\operatorname{Var} g$, is the number obtained by subtracting from $\operatorname{Var} g$ the sum of all the expressions

$$\|g(t)-g(t-0)\|+\|g(t+0)-g(t)\|-\|g(t+0)-g(t-0)\|$$

formed at points of discontinuity of g interior to I.

When A is a bounded linear functional on $C(E)$, there exists a function of bounded variation, $H: I \to E^*$, with $\|A\| = \operatorname{Var} H$ and $Af = \int_\alpha^\beta f(t)\,dH(t)$. This known theorem [Bochner and Taylor, Ann. of Math. (2) **39** (1938), 913–944; Gowurin, Fund. Math. **27** (1936), 235–268] is here proved in a different way, by reducing it back to the classical form of the theorem of F. Riesz. It is further shown that $\|A\| = \mathrm{R}\operatorname{Var} H$ for every H which generates the functional A by means of integration.

The principal contribution of this paper is the introduction and use of a concept of "function of Ω-variation" {the reviewer fails to see any special reason for the choice of the symbol Ω}. Consider a function $U: I \to L(E, F)$. Let $(t_0, \cdots, t_r)$ be any partition of I, and let $x_1, \cdots, x_r$ be any set of vectors in E, of norms not exceeding 1. If the expressions

$$V = \left\| \sum_{j=1}^{r} [U(t_j) - U(t_{j-1})]x_j \right\|$$

are bounded as the partition, the integer r, and the set $x_1, \cdots, x_r$ vary, the least upper bound of V is called $\Omega\operatorname{Var} U$, and U is said to be of bounded Ω-variation. When $F = B^*$, B being a Banach space, the concept of Ω-variation can be related to the ordinary concept of total variation as follows. If $z \in B$, $t \in I$, and $x \in E$, define $zU: I \to E^*$ by the formula $(zU)(t)x = [U(t)x]z$. Here $U(t) \in L(E, B^*)$ and $U(t)x \in B^*$. Then U is of bounded Ω-variation if and only if zU is of bounded variation (in the ordinary sense) for each z, and then

$$\Omega\operatorname{Var} U = \sup_{\|z\| \le 1} \operatorname{Var} zU.$$

The reduced Ω-variation of U is defined as $\mathrm{R}\Omega\operatorname{Var} U = \sup_{\|z\| \le 1} (\mathrm{R}\operatorname{Var} zU)$.

The following representation theorem is proved: If T is a bounded linear mapping of $C(E)$ into B^*, there exists $U: I \to L(E, B^*)$ such that U is of bounded Ω-variation, $\|T\| = \Omega\operatorname{Var} U = \mathrm{R}\Omega\operatorname{Var} U$, and $Tf = \int_\alpha^\beta f(t)\,dU(t)$ for each f in $C(E)$. The integral exists as a Riemann-Stieltjes integral. The extent to which U is uniquely determined by T is discussed.

Examples are given to show the distinctness of the concepts "of bounded variation" and "of bounded Ω-variation"; these examples also demonstrate some of the ways in which a function of bounded Ω-variation differs in its behavior from a function of bounded variation. *A. E. Taylor* (Los Angeles, Calif.)

7424:

Vidav, Ivan. Zur Axiomatik des Hilbertschen Raumes. Math. Z. **71** (1959), 458–460.

The note shows that in a system in which addition and multiplication by complex scalars are defined the usual properties of vector spaces follow from the existence of an inner product. *M. M. Day* (Urbana, Ill.)

7425:

Hopf, Eberhard. Zur Kennzeichnung der Euklidischen Norm. Math. Z. **72** (1959/60), 76–81.

The author considers a real vector space V with a Minkowski norm $|x|$. The latter satisfies the usual conditions except that $|x| = |-x|$ is not assumed. The definition of perpendicularity, $x \perp y$, is $|x - \alpha y| \geqq x$ for all α. The following theorem is proved: If there exists a function $f(\alpha, \beta)$ such that for all pairs (x, y) satisfying $x \perp y$ we have $|x+y| = f(|x|, |y|)$, then the norm in V is euclidean. The proof is based on three lemmas: (1) In a plane, the curve $|x| = 1$ contains no rectilinear segment; (2) $x \perp y$ implies $y \perp x$; (3) $|x| = |-x|$. The proofs are elementary. The case in which V has dimension $n = 2$ is not excluded as is the case in certain other characterizations of euclidean space. *E. R. Lorch* (New York, N.Y.)

7426:

Friedrichs, K. O.; and Shapiro, H. N. Integration over Hilbert space and outer extensions. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 336–338.

The notion of Gaussian distribution in n-dimensional space is extended to Hilbert space, yielding an integration theory of the finitely additive type which is rotation invariant. A method of extending Hilbert space to a larger set in order to produce a countably additive rotation-invariant measure is discussed.

J. T. Schwartz (Berkeley, Calif.)

7427:

Eisenstadt, B. J.; and Lorentz, G. G. Boolean rings and Banach lattices. Illinois J. Math. **3** (1959), 524–531.

Let X be a Banach lattice of measurable functions. If $\chi_e \in X$ is the characteristic function of a set e, $\Phi(e)=\|\chi_e\|$ is a function defined on a certain Boolean ring of sets. The authors attempt a characterization of norms on X by the functions Φ, and discuss especially in detail the problem of when a function Φ on a Boolean ring of sets B may be extended to a norm on some vector lattice X which includes B as characteristic functions. Most interesting is that for a continuous norm we have

$$\|x(t)+y(t)\| = \|x(t)\|+\|y(t)\|,$$

for two positive functions $x(t), y(t) \in X$ such that $x(t)-x(t')$ and $y(t)-y(t')$ have always the same sign, if and only if

$$\Phi(e_1 \cup e_2)+\Phi(e_1 \cap e_2) \leqq \Phi(e_1)+\Phi(e_2);$$

and then $\|x(t)\|=\int_0^\infty \Phi(e_\lambda)d\lambda$ for $e_\lambda=\{t:|x(t)| \leqq \lambda\}$.

H. Nakano (Sapporo)

7428:

Bear, H. S.; and Yood, Bertram. Multiplicative functionals on semigroups of continuous functions. Proc. Amer. Math. Soc. **10** (1959), 736–741.

For a compact Hausdorff space X, let $C(X)$ denote the multiplicative semigroup of all real-valued continuous functions on X. It was shown by A. N. Milgram [Duke Math. J. **16** (1949), 377–383; MR **10**, 612] that the topology of X is determined by the semigroup $C(X)$. When $C(X)$ is endowed with the topology of uniform convergence, let $\mathscr{F}(X)$ denote the set of all continuous, multiplicative, non-negative and non-constant functionals on $C(X)$. According to a result of A. Turowicz [Ann. Soc. Polon. Math. **20** (1947), 135–156; MR **10**, 127] and D. G. Bourgin [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 564–570; MR **12**, 421], for each $F \in \mathscr{F}(X)$, there exist a uniquely determined countable closed set $\{x_n\}$ in X and a summable sequence $\{a_n\}$ of positive numbers such that $F(f)=\prod |f(x_n)|^{a_n}$ for $f \in C(X)$. In the present paper, $\mathscr{F}(X)$ is regarded as a semigroup with multiplication defined pointwise, i.e., $(F \cdot G)(f)=F(f)G(f)$ for F, G in $\mathscr{F}(X)$ and $f \in C(X)$. For compact Hausdorff spaces X_1, X_2, it is shown that $\mathscr{F}(X_1)$ and $\mathscr{F}(X_2)$ are isomorphic if and only if there is a one-one mapping of X_1 onto X_2 preserving countable closed sets. Consequently, two compact Hausdorff spaces X_1, X_2 satisfying the first countability axiom are homeomorphic if and only if $\mathscr{F}(X_1)$ and $\mathscr{F}(X_2)$ are isomorphic. For compact Hausdorff spaces X_1, X_2 satisfying the first countability axiom, the following theorem is also proved. If σ is an isomorphism of $\mathscr{F}(X_1)$ onto $\mathscr{F}(X_2)$, and if $F_n(f) \to F(f)$ for all $f \in C(X_1)$ implies $\sigma(F_n)(g) \to \sigma(F)(g)$ for all $g \in C(X_2)$, then there exists a continuous isomorphism s of $C(X_2)$ onto $C(X_1)$ such that $\sigma(F)(g)=F(s(g))$ for all $F \in \mathscr{F}(X_1)$ and $g \in C(X_2)$. The proofs of these theorems make essential use of the Turowicz-Bourgin result stated above.

Ky Fan (Notre Dame, Ind.)

7429:

Horne, J. G., Jr. On the ideal structure of certain semirings and compactification of topological spaces. Trans. Amer. Math. Soc. **90** (1959), 408–430.

Let S be a set with a transitive relation R. A non-empty subset I is called an R-ideal if $f \in I$, $h \in S$ and hRf imply $h \in I$ and if $f, g \in I$ imply the existence of $e \in I$ such that fRe and gRe. In particular let S be a semigroup. Say e is a relative unit for f if e lies in the center of S and $fe=f$. The canonical order O on S is given by $O=\{(f,e): e \text{ is a relative unit for } f\}$. If S is the multiplicative semi-group $C(X)$ of all continuous functions on a compact space the O-ideals as just defined are the O-ideals of Milgram for $C(X)$ [Duke Math. J. **16** (1949), 377–383; MR **10**, 612]. If S is a lattice considered as a semigroup under $\wedge$, the O-ideals are the lattice ideals. A sub-semigroup S_0 of the semigroup S is called a Šilov (R) subset if fRe, eRg for elements f, e and g in S imply there exists e' in S_0 with fRe' and $e'Rg$. If $S=C(X)$ and $R=O$, the Šilov (O) subsets are the Šilov semigroups in $C(X)$ considered by Civin and Yood [Duke Math. J. **23** (1956), 325–334; MR **17**, 1227].

These definitions and examples along with others given by the author show how notions useful in the study of $C(X)$, where X is a topological space, and subsets of $C(X)$ can be advantageously considered in a more general setting. The notion of the Stone topology for maximal ideals in $C(X)$ is likewise considered anew. Let S be a family of subsets of a set T. For a subset A of S the dual Stone and Stone closures of A are defined respectively to be

$$\{M_0 \in S | M_0 \subset \bigcup \{M : M \in A\}\},$$

$$\{M_0 \in S | M_0 \supset \bigcap \{M : M \in A\}\}.$$

In general these define paratopologies rather than topologies. Criteria for compactness are obtained. An example shows that the maximal O-ideal space of a commutative ring with identity need not be compact in the dual Stone topology. The maximal ideals of a biregular ring with identity form a compact Hausdorff space in the dual Stone topology. Shirota [Osaka Math. J. **4** (1952), 121–132; MR **14**, 669] has considered R-lattices and proved some ideal structure theorems for them. Here a more general theory is obtained by studying R-semirings at an element. The requisite definitions are too lengthy for inclusion here. The concluding section deals with applications. We mention only one. Let X be a completely regular space. It is shown that the Stone-Čech compactification of X is characterized by any Šilov (O) subset of the multiplicative semigroup $K(X)$ of all continuous complex-valued functions on X.

B. Yood (Eugene, Ore.)

7430:

Edwards, R. E. Simultaneous analytic continuation. J. London Math. Soc. **34** (1959), 264–272.

The purpose of the author is to illustrate how a few general theorems about topological vector spaces may be profitably applied to the study of simultaneous analytic continuation of suitable families of holomorphic functions. The general result used as a tool in this paper, which subsumes Banach's principe de condensation des singularités and the so-called Gelfand lemma, appears as exercice 13, p. 36 in N. Bourbaki's *Espaces vectoriels topologiques*, chaps. III–V [Actual. Sci. Ind. no. 1229, Hermann, Paris, 1955; MR **17**, 1109]. The author gives, among other things, functional-analytic proofs of theorems 1, p. 84; 4, p. 87; and 5, p. 88 in S. Bochner and W. T. Martin, *Several complex variables* [Princeton Univ. Press, 1948; MR **10**, 366]. This application shows that, apart from the gain in generality and simplicity, the abstract approach permits the removal of superfluous hypotheses. This note deals also with the case of multivalent domains

and the corresponding concept of continuation, as well as the possibility of continuing functions initially defined on a relatively thin set. As a final comment, the author says that all theorems of his paper extend to other situations, once it is taken into account that the theory of analytic continuation rests almost entirely on one property of the sheaf of germs of holomorphic functions, namely, the property of being a separated space, and that there are several important known separated sheaves.

L. Nachbin (Rio de Janeiro)

7431:

Orlicz, W. Saks-space and its application in the theory of linear operators. Acta Math. Sinica **9** (1959), 143–149. (Chinese)

In this paper the author reviewed what has been done in the theory of Saks spaces since they were first studied by S. Saks in 1932 [see W. Orlicz, Studia Math. **11** (1950), 237–272; MR **12**, 418], together with a discussion of some unsolved problems and applications to the theory of linear operators. The results cited have appeared in the literature.

Choy-tak Taam (Washington, D.C.)

7432a:

U, Cun-sin'. The spaces $L_{M\Phi}$ and D_{Φ^λ}. I. Imbedding theorems. Sci. Record (N.S.) **3** (1959), 263–266. (Russian)

7432b:

U, Cun-sin'. The spaces $L_{M\Phi}$ and D_{Φ^λ}. II. Regularity, separability, and compact sets. Sci. Record (N.S.) **3** (1959), 267–269. (Russian)

If Δ is a measure space with measure μ, and $\Phi(u)$, $0 \leq u < +\infty$, is an increasing convex function, vanishing for $u=0$, let $M_\Phi(f)=\int_\Delta \Phi(|f|)d\mu$. Let $h(x)$ be an unbounded positive measurable function on Δ for which the sets of points x with $h(x) \leq n$ have finite measure. The space $L_{M\Phi}$ is a variant of the Orlicz space, it consists of all measurable functions f with $\|f\|_{L_{M\Phi}}=\inf k < +\infty$, where k is subject to the restriction $M_\Phi(k^{-1}f) \leq 1$. The space D_{Φ^λ} consists of all f with $\|f\|_{D_{\Phi^\lambda}}=\|f\|_{L_{M\Phi}}+\|hf\|_{L_{M\Phi}}<\infty$. Using methods of Luxemburg [Thesis, Technische Hogeschool te Delft, 1955; MR **17**, 285] and Ding [Sci. Record (N.S.) **1** (1957), 315–318; **2** (1958), 59–63; MR **20** #6653; **21** #2907] the author studies the following questions: conditions for the inclusion $D_{\Phi_2^\lambda} \subset L_{M\Phi_1}$; necessary and sufficient conditions for the separability or the regularity of D_{Φ^λ}; conditions for the compactness of subsets of $L_{M\Phi}$ and of D_{Φ^λ}.

G. G. Lorentz (Syracuse, N.Y.)

7433:

Krasnosel'skiĭ, M. A.; and Rutickiĭ, Ya. B. Orlicz spaces and non-linear integral equations. Trudy Moskov. Mat. Obšč. **7** (1958), 63–120. (Russian)

This is one of a series of papers on Orlicz spaces by the same authors. Some of the earlier papers in the series are listed and described briefly in the survey article by Krasnosel'skiĭ [Amer. Math. Soc. Transl. Ser. 2, Vol. 10, pp. 345–409, American Mathematical Society, Providence, R.I., 1958; MR **20** #1243. See also the authors' article, Voronež Gos. Univ. Trudy Sem. Funkcional. Anal. No. 2 (1956), 55–76; MR **18**, 811]. The motivation for this work lies in the problem of solving a nonlinear integral equation (1) $\varphi=\lambda A\varphi$, where (2) $A\varphi(x)=\int K(x,y)f[y,\varphi(y)]dy$, the domain G of integration being a compact subset of a Euclidean space, while the function f is required to be measurable in its first argument and continuous in its second. To deal with this equation one wishes to select a function space E such that A maps E into itself and is completely continuous. In cases where the function $f(x,u)$ grows too rapidly with $|u|$ the usual choices for E, such as $L^2(G)$ or $L^p(G)$, may not work. One of the main purposes of the authors is to show how equations (1) with "arbitrary" nonlinearities (but with suitable restrictions on the kernel) can be studied with the help of various classes of Orlicz spaces.

After the basic definitions are given for N'-functions $M(u)$ and their corresponding Orlicz spaces $L_M{}^*(G)$, four classes of these spaces are defined in terms of the growth properties of the related N'-functions. Besides the usual Δ_2 condition, which requires that $\limsup_{u\to\infty} M(2u)/M(u)$ be finite, the authors introduce the Δ' condition: $M(uv) \leq CM(u)M(v)$ for some positive C and all large u and v; the Δ^2 condition: $M(k_1u) \leq M^2(u) \leq M(k_2u)$ for some positive k_1, k_2 and all large u; and the Δ_3 condition: $M(k_1u) \leq uM(u) \leq M(k_2u)$. The first two conditions correspond to growth like a power while the last two permit exponential growth in $M(u)$.

Several theorems are obtained concerning conditions under which the operator (2) acts on an Orlicz space of a certain type, and is completely continuous. For example, as a special case of a more general theorem it is shown that if f satisfies the inequality (3) $|f(x,u)| \leq b+\exp(\alpha|u|)$ for all x and u, and if the kernel satisfies

$$\iint \exp[|K(x,y)|^{\beta_0}]dxdy < \infty \tag{4}$$

for some $\beta_0>1$, then the operator (2) is completely continuous on the space $L_{M_\beta}{}^*$, where $M_\beta(u)=\exp(|u|^\beta)-1$, providing that $1<\beta<\beta_0$.

The authors next show how this type of result, together with the Schauder fixed point theorem, may be used to obtain existence theorems for equation (1) under various conditions. They also take up the case of positive definite symmetric kernels. In order to use the variational methods of Golomb-Vaĭnberg-Rothe, it is necessary to consider first the decomposition of the linear operator

$$K\varphi = \int K(x,y)\varphi(y)dy$$

acting on one Orlicz space to another, in the form $K=HH^*$. Here H^* acts on $L_\Psi{}^*$ to L^2 while H acts on L^2 to $L_\Phi{}^*$, where $\Phi(u)$ and $\Psi(v)$ are complementary N'-functions and where $\Psi(v)$ satisfies the Δ_2 condition. [See the Krasnosel'skiĭ survey article cited above, p. 363.]

One of the theorems obtained on decomposition is a generalization to Orlicz spaces of a theorem due to Krasnosel'skiĭ for operators on the space L^p to L^q [Dokl. Akad. Nauk. SSSR **82** (1952), 333–336; MR **13**, 661].

The decomposition theorems are then used in combination with known results to obtain an existence theorem for the equation (1) when the kernel K is symmetric, positive definite and satisfies the inequality (4), while the function $f(x,u)$ satisfies (3). The paper ends with a brief discussion of a non-linear eigenvalue problem involving bifurcation.

D. H. Hyers (Los Angeles, Calif.)

7434:

Henriksen, Melvin. Multiplicative summability methods and the Stone-Čech compactification. Math. Z. **71** (1959), 427–435.

A summability method φ is a linear functional on a linear subspace, denoted $S(\varphi)$, of $C(N)$, the space of all real sequences. If A is a (countably infinite) matrix, then φ_A denotes the corresponding summability method; if B is obtained from A by deleting some rows, then φ_B is called a submethod of φ_A. Assuming the continuum hypothesis, it is proved that, for any regular φ_A, there is a "simultaneously consistent" (i.e., such that $\sum_1^n \varphi_i(f_i)=0$ whenever $\varphi_i \in \Phi$, $f_i \in S(\varphi_i)$, $\sum f_i=0$) family Φ of submethods of φ_A such that $S(\varphi)$, $\varphi \in \Phi$, cover $C^*(N)$ (the space of all bounded real sequences). If the restriction (denoted φ^* here) of φ to $S^*(\varphi)=S(\varphi) \cap C^*(N)$ is multiplicative, then φ is called b-multiplicative. For any method φ, let $F(\varphi)$ denote the set of all $x \in \beta N-N$ such that $f \in S^*(\varphi)$ implies $f(x)=\varphi(f)$. It is proved that the correspondence $\varphi^* \to F(\varphi)$, φ regular b-multiplicative, is one-to-one onto all closed non-void $F \subset \beta N-N$. By means of this result, the existence is shown of a regular b-multiplicative φ_A which is not a submethod of φ_I. The problem is stated of determining which closed $F \subset \beta N-N$ are of the form $F=F(\varphi_A)$, φ_A a regular b-multiplicative matrix summability method. *M. Katětov* (Prague)

7435:

Ivanov, V. V.; and Simonenko, I. B. Approximate search for all solutions of a given linear equation in Banach space. Dokl. Akad. Nauk SSSR **126** (1959), 1172–1175. (Russian)

Let E_1 and E_2 be complex Banach spaces with basis, and let K be a linear operator mapping E_1 into E_2. The null-space of K is supposed to have dimension α while the null-space of K^*, the operator adjoint to K, has dimension β. Then a linear subspace E_1^0 and elements $u_j \in E_2$ $(j=1, 2, \cdots, \beta)$ may be found such that the equation

$$Kx^0+\sum_{j=1}^{\beta} c_j u_j = y \tag{1}$$

has for every $y \in E_2$ one and only one solution

$$\chi = \{x^0, c_1, c_2, \cdots, c_\beta\},$$

where $x^0 \in E_1^0$ and where the c_j are complex numbers.

Let now P_N be a sequence of projection operators on E_2. It is supposed that, from a certain N on, the range M_N of P_N contains $u_1, u_2, \cdots, u_\beta$. The object is to find to the solution χ of (1) approximations of the form

$$\chi_n = \{\varphi_n, c_1, c_2, \cdots, c_\beta\},$$

where $\varphi_n \in L_n$, a linear subspace of E_1^0 of dimension $n=N-\beta$. It follows from results of Pol'skiĭ [same Dokl. **111** (1956), 1181–1184; MR **18**, 802] that for the convergence of the approximations χ_n to the solution χ of (1) it is sufficient that for some positive constant C

$$\|P_N \tilde{K}\chi_n\| \geq C\|\tilde{K}\chi_n\|, \tag{2}$$

where $\tilde{K}$ denotes the operator given by the left member of (1). The authors prove (Theorem 2) that (2) is implied by the following condition:

$$\|P_N K\varphi_n - K\varphi_n\| \leq \varepsilon_N \|\varphi_N\|, \quad \lim_{N\to\infty} \varepsilon_n = 0. \tag{3}$$

The authors' method is then based on the possibility of defining the family $\{P_N\}$ of projections in such a way that (3) is satisfied. Two examples illustrate the procedure. *E. H. Rothe* (Ann Arbor, Mich.)

7436:

Sz.-Nagy, Béla. Completely continuous operators with uniformly bounded iterates. Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 89–93. (Hungarian and Russian summaries)

Let T be a linear operator in a real or complex Hilbert space such that $\|T^n\| \leq M$ $(n=0, 1, 2, \cdots)$. The author conjectures that under this assumption T must be similar to a contraction, i.e., there exists a bounded selfadjoint operator Q with bounded inverse such that $\|QTQ^{-1}\| \leq 1$. In the present paper this is proved for the case of completely continuous T in a complex Hilbert space. If the spectral radius of T is less than 1, then Q can be simply taken as the positive square root of $\sum_{k=0}^{\infty} T^{*k}T^k$. If T has eigenvalues of absolute value 1, the corresponding subspaces are of finite dimension, and the situation can still be handled without much difficulty. *A. Kordnyi* (Cambridge, Mass.)

7437:

Gehér, László. Über ähnliche lineare Transformationen in endlichdimensionalen Räumen. Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 95–99. (Hungarian and Russian summaries)

The conjecture stated in the paper reviewed above is meaningful for any normed linear space. In this paper the author proves it for the case of finite-dimensional l^p-spaces, making use of the Jordan canonical form of matrices. *A. Korányi* (Cambridge, Mass.)

7438:

Brodskiĭ, M. S. Integral representations of bounded non-selfadjoint operators with a real spectrum. Dokl. Akad. Nauk SSSR **126** (1959), 1166–1169. (Russian)

A bounded linear operator on a Hilbert space H is here called simple if there is no proper reducing subspace of H on which A is selfadjoint. A simple operator A is in class Ω_R if it satisfies: (1) A has only real spectrum; (2) $V=i(A^*-A)$ is compact; and (3) the sum of the absolute values of the eigenvalues of V (counting multiplicities) is a number $L<\infty$. Let $A \in \Omega_R$, let $r=\dim VH^-$ $(r \leq \infty)$, and consider the space L_2^r, the direct sum of r copies of $L_2[0, L]$. The elements of L_2^r are regarded as row matrices of functions in $L_2[0, L]$. Then there is a Hilbert space H_0 and a bounded selfadjoint operator A_0 on H_0 such that $\tilde{A}=A \oplus A_0$ is unitarily equivalent with an operator $\tilde{B}$ on L_2^r of the form

$$\tilde{B}f(x) = \alpha(x)f(x)+i\int_x^L f(t)\Pi(t)dtJ\Pi^*(x),$$

where (a) α is a bounded monotone increasing function on $[0, L]$, (b) $\Pi(t)=(\pi_{ij}(t))$ is an $r \times r$ matrix of functions on $[0, L]$ satisfying

$$\operatorname{tr} \Pi^*(x)\Pi(x) \equiv 1; \quad \int_0^L \Pi^*(x)\Pi(x)dx = \operatorname{diag}$$

and (c) $J=\operatorname{diag}(\pm 1)$. This theorem of Livšic [Dokl. Akad. Nauk SSSR **84** (1952), 873–876; MR **14**, 184] provides a triangular model for $\tilde{A}$, here called a simple extension of A.

Now the author introduces the resolution of unity $F(x)$, $0 \leq x \leq L$, on L_2^r

$$F(x)h(t) = h(t), \quad 0 \leq t \leq x,$$
$$= 0, \quad x < t \leq L,$$

and shows that $\tilde{B} = \int_0^L \alpha(x)dF(x) + i\int_0^L F(x)\tilde{W}dF(x)$ with $\tilde{W} = i(\tilde{B}^* - \tilde{B})$. Consequently, if U is the isomorphism such that $\tilde{A} = U^{-1}\tilde{B}U$ and if $E(x)$ is defined as $U^{-1}F(x)U$ then $E(x)$ is also a resolution of unity (on $H \oplus H_0$) and

$$\tilde{A} = \int_0^L \alpha(x)dE(x) + i\int_0^L E(x)\tilde{V}dE(x),$$

where $\tilde{V} = i(\tilde{A}^* - \tilde{A})$. It is also shown that

$$\operatorname{tr}(|V|^{1/2}E(x)|V|^{1/2}) \equiv x, \quad 0 \leq x \leq L.$$

If now P denotes the projection of $H \oplus H_0$ onto H and if $E_0(x) = PE(x)P|H$, then a triangular model

$$A = \int_0^L \alpha(x)dE_0(x) + i\int_0^L E_0(x)VdE_0(x)$$

for the original operator A is obtained. As compressions of projections the operators $E_0(X)$ do not in general comprise a resolution of unity. The author shows, however, that they are in fact projections in the special case spec $A = 0$ and $V > 0$. *A. Brown* (Houston, Tex.)

7439:

Martynyuk, A. E. On a generalization of a variational method. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 374–377. (Russian)

A positive operator, A, in the generalized sense is a linear operator over a domain, $D(A)$, dense in a Hilbert space, which admits the factorization $A = A_0A_1$, where A_0 is self-adjoint and A_1 is closed and for every non-zero element u in $D(A)$, $(Au, A_1u) > 0$. If, then, A is such an operator, $\{u|Au = f\}$ is equivalent to $\{u|u$ minimizes $\Phi = (Au, A_1u) - (A_1u, f) - (f, A_1u)\}$. Some results on minimizing sequences (for Φ) are given. *D. G. Bourgin* (Urbana, Ill.)

7440:

Brownell, F. H. Eigenvector approximation for bounded Hermitian operators on Hilbert space. Univ. Nac. Tucumán. Rev. Ser. A **12**, 57–71 (1959).

Let H be a bounded hermitian operator on a Hilbert space, u_0 a normalized "trial vector"; the paper deals primarily with methods of approximating to an eigenvalue λ, such that a corresponding normalized eigenvector x_1 is near u_0 in norm. (In this review, assume that $(Hu_0, u_0) = 0$, and that $u_0, Hu_0, H^2u_0, \cdots$ span the space.) The central idea, as in earlier papers by Kato and others which the author cites, is that if $p(\lambda)$ is a real polynomial $p(H)u_0$ can be found, and its normalization u will be close to x_1 if $|p(\lambda)|$ is much larger at $\lambda = \lambda_1$ than on the rest of the spectrum. In the principal theorem, however, $p(\lambda) = \exp(-p^2\lambda)$, so that u is not necessarily calculable. The hypotheses are bounds on $\|H\|$, on the distance from λ_1 to the rest of the spectrum, and on $\|u_0 - x_1\|$; the conclusion is a bound on $\|u - x_1\|/\|u_0 - x_1\|$. In another theorem, with similar hypotheses, a different approximation procedure is used. Let P be the projection onto $\{u_0\}^\perp$. Then the (not normalized) approximation to x_1 is $u_0 - (PHP)^{-1}PHu_0$; the inverse is proved to exist on range (P); a bound on the norm of the error term is given which is quadratic in $\|u_0 - x_1\|$. *C. Davis* (Providence, R.I.)

7441:

Fixman, Uri. Problems in spectral operators. Pacific J. Math. **9** (1959), 1029–1051.

Restrictions to an invariant subspace of operators having a resolution of the identity in the sense of Dunford (spectral operators) are discussed, and the results obtained are applied to show that various unitary operators in the classes l_p and C are not spectral. Conditions on the norm of rational functions of an operator necessary and sufficient in order that the operator be spectral and of scalar type are given. *J. T. Schwartz* (Berkeley, Calif.)

7442:

McCarthy, Charles A. The nilpotent part of a spectral operator. Pacific J. Math. **9** (1959), 1223–1231.

Let T be a bounded spectral operator on a complex Banach space; let N be its quasinilpotent part and $E(\sigma)$ (σ a Borel set) its resolution of the identity. The context is that of Dunford's theory [Pacific J. Math. **4** (1954), 321–354; MR **16**, 142]. It is assumed that there is an integer m and a constant K such that

$$|R(\lambda, T)E(\sigma)| \leq Kd(\lambda, \sigma)^{-m}$$

if λ is not in $\tilde{\sigma}$ and $|\lambda| \leq |T| + 1$. Here $d(\lambda, \sigma)$ is the distance from λ to σ. It was shown in the cited paper of Dunford that this growth condition of mth order implies $N^m = 0$ if the underlying space is a Hilbert space. This paper shows that the situation is not the same in general for Banach spaces. Theorem: If for some p the Hausdorff p-measure of σ is 0, then $N^kE(\sigma) = 0$ when k is an integer $\geq p + m - 1$. As a corollary, $N^{m+2} = 0$. An example is given to show that this is the best that can be said in general. If the space is separable or weakly complete, $N^{m+1} = 0$. This result also is the best possible. {There is a misprint on the second line of page 1231; for $N^m = 0$ read $N^m \neq 0$.} There are special conditions on σ (involving Hausdorff measure of dimensions 1 and 2) under which it is sure that $N^kE(\sigma) = 0$ for $k = m$ or $m + 1$. *A. E. Taylor* (Los Angeles, Calif.)

7443:

Bade, William G. A multiplicity theory for Boolean algebras of projections in Banach spaces. Trans. Amer. Math. Soc. **92** (1959), 508–530.

In the last years many contributions have been given to the generalization in Banach spaces of the theory of commutative W^* algebras in Hilbert space. The present article constitutes a new important step in this direction, concerning especially the multiplicity theory.

Let $\mathfrak{B}$ be a complete (in the author's sense) Boolean algebra (B.A.) of projections in a real or complex Banach space $\mathfrak{X}$. Firstly, it is shown that there exists a unique multiplicity function m defined on $\mathfrak{B}$ (i.e., the values of m are cardinals, $m(0) = 0$ and $m(\vee_\alpha E_\alpha) = \vee_\alpha m(E_\alpha)$, $\{E_\alpha\} \subseteq \mathfrak{B}$) such that, for those $E \in \mathfrak{B}$ for which every family of disjoint projections ($\in \mathfrak{B}$, $\leq E$) is at most countable, $m(E)$ is the smallest cardinal power of a set of cyclic subspaces $\mathfrak{M}(x) = \operatorname{clm}\{Ax; A \in \mathfrak{B}\}$ spanning the range of E; by a result concerning complete abstract B.A., it is deduced

that there is a unique decomposition of the identity $I=\vee_n E_n$ into disjoint projections such that each $E_n\neq 0$ has uniform multiplicity n (i.e., if $0\neq G\leq E$, $G\in\mathfrak{B}$, then $m(G)=m(E)$).

Let Ω be the Stone space of $\mathfrak{B}$, $\mathfrak{B}$ be considered as a spectral measure on Ω and f be a Borel function defined on Ω. Defining $S(f)$ by

$$S(f)y=\lim\int_{e_m} f(\omega)E(d\omega)y \quad (e_m=\{\omega;\,|f(\omega)|\leq m\}),$$

for those $y\in D(S(f))=\{y;\ \lim\int_{e_m} f(\omega)E(d\omega)y \text{ exists}\}$, the author proves the fundamental relation

$$\mathfrak{M}(x)=\{S(f)x;\,x\in D(S(f))\}.$$

For $E\in\mathfrak{B}$ of finite uniform multiplicity n it is given a representation in the direct sum $\sum_1^n L_1(\Omega,\mu_i)$, where $\mu_i=x_i^*E(\cdot)x_i$ $(i=1,2,\cdots,n)$ are positive bounded measures.

Similar facts are established for the adjoint B.A. $\mathfrak{B}^*$ in $\mathfrak{X}^*$. Using the above mentioned representation, it is shown that $E\in\mathfrak{B}$ has finite uniform multiplicity n if and only if $E^*\in\mathfrak{B}^*$ has finite uniform multiplicity n, so that if $\mathfrak{X}$ is separable, $m(E)=m(E^*)$, $E\in\mathfrak{B}$.

As application is given a fact concerning a similarity relation between scalar type spectral operators in separable Banach spaces, whose resolution of the identity forms a B.A. containing no projection of infinite uniform multiplicity, and normal operators in Hilbert space; this similarity, by means of a closed densely defined linear map with densely defined inverse, conserves the invariant subspaces and the functional calculus.

C. Foiaş (Bucharest)

7444:

Nachbin, Leopoldo. Algebras of finite differential order and the operational calculus. Ann. of Math. (2) **70** (1959), 413–437.

This paper contains a detailed account of results announced previously in a paper entitled "On the operational calculus with differentiable functions" [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 698–700; MR **20** #4610].

C. E. Rickart (New Haven, Conn.)

7445:

Miyadera, Isao. Semi-groups of operators in Fréchet space and applications to partial differential equations. Tôhoku Math. J. (2) **11** (1959), 162–183.

A Fréchet space is a topological vector space which is locally convex, complete and metrizable. The author considers semi-groups of operators $\{T(\xi);\,0\leq\xi<\infty\}$ where, for each $\xi\geq 0$, $T(\xi)$ is a continuous operator from the Fréchet space X into itself such that: (1) $T(\xi+\eta)=T(\xi)T(\eta)$, $T(0)=I$; (2) there exists $\sigma>0$ such that $\{e^{-\sigma\xi}T(\xi)x;\,\xi\geq 0\}$ is bounded in X for each $x\in X$; and (3) $T(0^-)=T(0)$. (In a Banach space (2) is a consequence of (1) and (3).) For semi-groups with the above properties the Hille-Yosida theorem holds—there exists an infinitesimal generator A which is a closed linear operator with dense domain in X, the resolvent $R(\lambda,A)$ exists for each $\lambda>\sigma$ and the set $\{[(\lambda-\sigma)R(\lambda,A)]^n x;\,\lambda>\sigma,\,n=0,1,2,\cdots\}$ is bounded. The operators $T(\xi)$ satisfy the relation

$$(*)\quad T(\xi)x=\lim_{\lambda}\exp(-\lambda\xi)\sum_k(\lambda\xi)^k(R(\lambda,A))^k/k!\cdot x.$$

Conversely, for a closed operator A satisfying the above conditions the equation (*) defines a semi-group having A as infinitesimal generator.

The author applies this result to obtain existence and uniqueness of regular C^∞ solutions of the initial value problem for parabolic equations in E_m. The Fréchet space in this case consists of the C^∞ functions in E_m all of whose derivatives belong to L_2, and the topology is determined by the set of semi-norms $p_\alpha^2(\varphi)=\int_{E_m}|D^\alpha\varphi|^2dx$.

A. N. Milgram (Berkeley, Calif.)

7446:

Trotter, H. F. On the product of semi-groups of operators. Proc. Amer. Math. Soc. **10** (1959), 545–551.

Let Ω and Ω' be two operators on the Banach space X each of which is the infinitesimal generator of a strongly continuous semi-group of bounded linear operators. The present paper considers the question of when the operator $\Omega+a\Omega'$ (or its closure) is also an infinitesimal generator for positive values of a. In particular, it is shown that under suitable conditions the set of such a is an open set in $[0,\infty)$ and contains a neighborhood of 0. One of the conditions imposed is to the effect that the norms of all semi-groups under study must be bounded by a function of the form $e^{\omega t}$ instead of the usual $Me^{\omega t}$. Of some interest, then, is an example in the present paper which shows that X cannot be renormed so that two given semi-groups will simultaneously satisfy this condition.

G. Hufford (Seattle, Wash.)

7447:

Krasnosel'skiĭ, M. A.; and Sobolevskiĭ, P. E. Fractional powers of operators acting in Banach spaces. Dokl. Akad. Nauk SSSR **129** (1959), 499–502. (Russian)

Let E be a (complex) Banach space. The authors announce, without proof, several theorems regarding fractional powers of certain members of the class $\mathfrak{A}(E)=\{\text{all mappings into } E \text{ of a dense subset of } E\}$. An operator A is called "normally-positive" if $A\in\mathfrak{A}(E)$ and if there exists $k\geq 0$ such that $\|(A+tI)^{-1}\|\leq k(1+t)^{-1}$ for all $t\geq 0$ (this definition also requires that, for any $t\geq 0$, the transformation $(A+tI)^{-1}$ be a bounded operator defined on the whole space E). If E is a Hilbert space, then a normal operator $A\in\mathfrak{A}(E)$ will be normally-positive if the negative axis $\{\lambda:\,\lambda\leq 0\}$ is interior to the resolvent set of A. In the case $E=L^p$, Solomyak [Dokl. Akad. Nauk SSSR **122** (1958), 766–769; MR **21** #3775] has proved that certain differential operators are normally-positive. It is implied henceforth that A is normally-positive and that $A\in\mathfrak{A}(E)$. If $\alpha>0$, let n be the smallest integer such that $n>\alpha-1$; the authors write: $A^{-\alpha}=k_\alpha\int_0^\infty t^{n-\alpha}(A+tI)^{-n-1}dt$, where

$$k_\alpha=\frac{\sin\pi\alpha}{\alpha}\,\frac{n!}{(1-\alpha)(2-\alpha)\cdots(n-\alpha)}$$

in case α is not an integer (otherwise $k_\alpha=\lim_{\lambda\to\alpha}k_\lambda$). If $\alpha=1$ then $AA^{-\alpha}=I$. Theorem: $\{A^{-\alpha}:\,\alpha>0\}$ forms a strongly continuous semi-group of bounded operators on E. If $\alpha>0$ and $A^{-\alpha}x=0$ with $x\in E$, then $x=0$. The range $\mathfrak{R}(A^{-\alpha})$ is a dense subset of E. In case $\alpha\geq\beta>0$ then $\mathfrak{R}(A^{-\alpha})\subset\mathfrak{R}(A^{-\beta})$. The preceding results make it possible to define A^λ for any real value of λ. Next, suppose $\alpha\cdot\beta>0$; then $A^\alpha A^\beta=A^{\alpha+\beta}$, and the relation

$$\|A^\alpha x\|\leq k\cdot\|A^\beta x\|^{\alpha/\beta}\|x\|^{1-\alpha/\beta}$$

holds whenever $|\alpha| \leq |\beta|$ and for any x in the domain of A^β (where k is independent of x). Now follows one out of several "subjugation" theorems; by definition, F is said to be "subjugated" to T if F and T are such members of $\mathfrak{A}(E)$ that the domain of F includes the domain of T, while $\|Fx\| \leq k \cdot \|Tx\|$ for all x in the domain of T (where k is independent of x). Theorem: If F is subjugated to A^α and $\beta > \alpha$, then $\|Fx\| \leq k \cdot \|A^\beta x\|^{\alpha/\beta} \cdot \|x\|^{1-\alpha/\beta}$ whenever x is in the domain of A^β (where k is independent of x). Theorem: Suppose that both A and B are normally-positive members of $\mathfrak{A}(E)$, and suppose moreover that A^r is subjugated to B^s, where $r > 0$ and $s > 0$: if $0 \leq \alpha \leq \beta \leq 1$, then $A^{\alpha r}$ is subjugated to $B^{\beta s}$. The authors conclude by generalizing a theorem of Erhard Heinz [Math. Ann. 415–438; MR **13**, 471; cf. p. 420].

G. L. Krabbe (Lafayette, Ind.)

7448:

Simon, Arthur B. On the maximality of vanishing algebras. Amer. J. Math. **81** (1959), 613–616.

Let G be a locally compact abelian group, μ the Haar measure of G, L^1 the usual convolution algebra, S a subset of G and L_S the set of all functions $f \in L^1$ which vanish a.e. on the complement of S. If G is the real line and $S = [0, \infty)$, then, as was proved by J. Wermer, L_S is maximal. In this paper the author proves the following (converse) theorem. Let S be a measurable subsemigroup of G. If L_S is a maximal proper closed subalgebra of L^1, then G is (continuously isomorphic with) either a discrete subgroup of the reals or the real line itself (with S mapping onto the non-negative part). The proof is based on a theorem concerning Hölder groups which is due to F. B. Wright [Duke Math. J. **24** (1957), 567–571; MR **19**, 967] and on the following results. Let H be a measurable semigroup contained in G such that L_H is a maximal proper closed subalgebra of L^1. Then (1) $H \cap (-H) = \{0\}$; (2) there is a maximal subsemigroup F_0 of the group G such that $L_{F_0} = L_H$ and $F_0 \cap (-F_0) = \{0\}$.

C. Ionescu Tulcea (New Haven, Conn.)

7449:

Luchins, Edith H. On strictly semi-simple Banach algebras. Pacific J. Math. **9** (1959), 551–554.

The author defines the "strict radical" of an algebra to be the intersection of those of its 2-sided ideals which are regular maximal right ideals. If the strict radical is zero, the algebra is called strictly semi-simple (sss). An example of a sss Banach algebra is the real algebra $C(X, Q)$ of all quaternion-valued functions which are continuous and vanish at infinity on the locally compact Hausdorff space X. Also, any subalgebra of $C(X, Q)$ is sss. It is proved that a real Banach algebra is sss if and only if it is isomorphic with a subalgebra of $C(X, Q)$. Thus any subalgebra (closed or not) of a sss real Banach algebra is sss. Finally it is proved that the strict radical of a real Banach algebra contains the set of topologically nilpotent elements.

C. E. Rickart (New Haven, Conn.)

7450:

Marinescu, G. Algèbres localement convexes de type φ. Rev. Math. Pures Appl. **3** (1958), 87–91.

A linear algebra E is of the type discussed here if it has a family $\{|\cdot|_{a,\varphi(a)} : a \in A\}$ of seminorms, where A is a set and $\varphi : A \to A$ such that (1) $\varphi(\varphi(a)) = \varphi(a)$, (2) for $a, a' \in A$ there is an $a'' \in A$ such that $|x|_{a,\varphi(a)}$ and $|x|_{a',\varphi(a')}$ do not exceed $|x|_{a'',\varphi(a'')}$, (3) $|xy|_{a,\varphi(a)} \leq |x|_{a,\varphi(a)} |y|_{\varphi(a),\varphi(a)}$, and (4) $|1|_{\varphi(a),\varphi(a)} = 1$. It is shown that inversion can be studied by a Neumann series technique in these algebras.

R. Arens (Los Angeles, Calif.)

7451:

Foguel, S. R. Boolean algebras of projections of finite multiplicity. Pacific J. Math. **9** (1959), 681–693.

Generalising his earlier results on operators in Hilbert space, the author applies Bade's analysis of multiplicity theory for Boolean algebras of projections in Banach space [#7443 above] to show that the algebra of operators commuting with an algebra of projections of uniform multiplicity may be represented by matrices of measurable functions. This representation is then used to give results on convergence, on generalised nilpotents, compact operators, etc., lying in such an algebra.

J. T. Schwartz (Berkeley, Calif.)

7452:

Saitô, Teishirô. The direct product and the crossed product of rings of operators. Tôhoku Math. J. (2) **11** (1959), 299–304.

N. Suzuki has defined a crossed product (M, G) of a finite factor M and a group of outer automorphisms G [Tôhoku Math. J. (2) **11** (1959), 113–124; MR **21** #4363]. In this paper, the author proves that the direct product $(M_1, G_1) \otimes (M_2, G_2)$ is isomorphic with the crossed product $(M_1 \otimes M_2, G_1 \otimes G_2)$, where the M_i's are finite factors with invariants $C = 1$. The author also proves the following theorem. Let M be the approximately finite factor on a separable Hilbert space and G_0 an arbitrary countable group; then there exists a group G of automorphisms of M, isomorphic to G_0, such that the crossed product (M, G) is a factor of type II_1 with the property Γ [F. J. Murray and J. von Neumann, Ann. of Math. (2) **44** (1943), 716–808; MR **5**, 101].

Ti Yen (Chicago, Ill.)

7453:

Tomiyama, Jun. On the product projection of norm one in the direct product of operator algebras. Tôhoku Math. J. (2) **11** (1959), 305–313.

"... for two W^*-algebras M_1 and M_2 and their W^*-subalgebras N_1 and N_2, if π_1 and π_2 are normal projections of norm one from M_1 and M_2 to N_1 and N_2 respectively, there exists the unique normal projection of norm one π from $M_1 \otimes M_2$ to its W^*-subalgebra $N_1 \otimes N_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$ and a similar result holds for the projections of norm one in C^*-algebras." (From the author's introduction)

S. Sherman (Philadelphia, Pa.)

7454:

Nakamura, Masahiro; and Takeda, Zirô. On the extensions of finite factors. I. Proc. Japan Acad. **35** (1959), 149–154.

This paper studies factor sets associated with crossed-product constructions of von Neumann factors. Factorization of subalgebras is discussed and the product is shown to satisfy the usual elementary properties of crossed products.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

7455:

Hongo, Eishi. On some properties on quasi-unitary algebras. Bull. Kyushu Inst. Tech. Math. Nat. Sci. **4** (1958), 1–6.

In a previous paper [same Bull. **3** (1957), 1–10; MR **19**, 872], the author associated with a quasi-unitary algebra having a semi-finite left ring a certain unitary algebra. In the present paper, this result is used to obtain some results concerning the structure of the given quasi-unitary algebra. For example, a necessary and sufficient condition for the left ring to be of finite class is given.

C. E. Rickart (New Haven, Conn.)

7456:

Hongo, Eishi. On left multiplicative operators on a quasi-unitary algebra. Bull. Kyushu Inst. Tech. Math. Nat. Sci. **5** (1959), 19–22.

This paper is a continuation of two previous papers by the author [same Bull. **3** (1957), 1–10; MR **19**, 872, and paper reviewed above]. It contains proofs of some theorems obtained by Pukánszky [Acta Sci. Univ. Szeged. **16** (1955), 103–121; MR **17**, 515].

C. E. Rickart (New Haven, Conn.)

7457:

Granas, A. Sur la notion du degré topologique pour une certaine classe de transformations multivalentes dans les espaces de Banach. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 191–194. (Russian summary, unbound insert)

The notion of the topological mapping degree, originally defined in Euclidean n-space, has been generalized for mappings in a Banach space E [Leray and Schauder, Ann. École Norm. Sup. **51** (1934), 45–78]. The same is true for the notion of the characteristic of a vector field. In the present paper the author generalizes these notions for "multiple-valued" mappings $\varphi(x)$ defined for points x of some subset A of the Banach space E into the set of subsets of E. He assumes: (i) φ is "upper semicontinuous", i.e., φ has the property that if $\{x_n\}$ is a sequence of points in A which has a limit $x \in A$, if $\{y_n\}$ is a sequence such that $y_n \in \varphi(x_n)$, and if $y = \lim_{n\to\infty} y_n$ exists, then $y \in \varphi(x)$; (ii) for each $x \in A$, the set $\varphi(x)$ is convex; (iii) $\varphi(x) - x$ is completely continuous, i.e., $\bigcup_{x\in A}(\varphi(x)-x)$ is relative compact in E.

The construction of the desired definitions involves the following steps. (a) If $E = E^n$, the Euclidean space, the characteristic $\chi(\varphi)$ of a non-vanishing vector field on a sphere $S^{n-1} \subset E^n$ is defined provided φ satisfies the above assumptions (i) and (ii). (b) By an approximation procedure reminiscent of the one used by Leray and Schauder ("going to the limit in the dimension") the author succeeds in defining $\chi(\varphi)$ for a non-vanishing vector field on a sphere S in Banach space provided that also assumption (iii) is satisfied. (c) If φ is defined on the ball V bounded by S and if φ satisfies also assumption (iii) then the mapping degree is defined in terms of χ.

The definition of $\chi(\varphi)$ in E^n (step (a)) is as follows. Let $\Gamma = \Gamma_\varphi$ be the graph of φ, i.e., the set of couples $z = (x, y)$ with $x \in S^{n-1}$, $y \in \varphi(x)$. For $z = (x, y) \in \Gamma$, define the projections $r(z) = x$ and $s(z) = y$. Let $t(z) = s(z)/\|s(z)\|$ such that $t(z) \in S_0{}^{n-1}$, the unit sphere in E^n. The projection r induces a mapping r^* of the homology group of dimension $(n-1)$ of Γ into that of S^{n-1}. It follows from the assumptions made about φ that the "generalized" Vietoris mapping theorem [E. G. Begle, Ann. of Math. (2) **51** (1950), 534–543; MR **11**, 677] may be applied. Consequently r^* is an isomorphism onto such that $(r^*)^{-1}$ exists, and $t^*(r^*)^{-1}$ is a mapping of the homology group H^{n-1} of dimension $n-1$ of S^{n-1} into $H_0{}^{n-1}$, the homology group of dimension $(n-1)$ of $S_0{}^{n-1}$, where t^* is defined with respect to t as r^* with respect to r. If then u, u_0 are generators for H^{n-1} and $H_0{}^{n-1}$, respectively, then the image of u under the map $t^*(r^*)^{-1}$ is an integer multiple of u_0. This multiple is defined as $\chi(\varphi)$.

Some of the basic properties of $\chi(\varphi)$ and of the mapping degree are proved.

E. H. Rothe (Ann Arbor, Mich.)

7458:

Krasnosel'skiĭ, M. A.; Rutickiĭ, Ya. B.; and Sultanov, R. M. A non-linear operator acting on spaces of abstract functions. Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Mat. Teh. Nauk **1959**, no. 3, 15–21. (Russian. Azerbaijani summary)

Let Ω be a closed set in euclidean space and let B be a Banach space. A linear space $\bar{B}$ of functions from Ω into B (modulo null functions) is called a "χ-space" if (1) $\bar{B}$ is provided with a norm such that convergence in norm implies convergence in measure; (2) the constants belong to $\bar{B}$; (3) if χ_E is the characteristic function of a measurable set $E \subset \Omega$ and $u \in \bar{B}$, then $\chi_E u \in \bar{B}$ and $\|\chi_E u\| \le \|u\|$; (4) in case Ω has infinite measure, there is a sequence of subsets Ω_n of finite measure whose union is Ω and such that $\lim \|\chi_{\Omega_n} u\| = \|u\|$ for all $u \in \bar{B}$. Let f be a mapping from $\Omega \times B$ into a Banach space B_1, and assume that $f(t, x)$ is strongly measurable in t for each fixed $x \in B$ and strongly continuous in x for almost all $t \in \Omega$. With f is associated an operator $\mathfrak{f}$ from $\bar{B}$ into an analogous space $\bar{B}_1$: $\mathfrak{f}u(t) = f(t, u(t))$. Theorem: Suppose that for each $u \in \bar{B}$, $\|\chi_E \mathfrak{f} u\| \to 0$ as the measure of the set $E \subset \Omega$ approaches 0. Then $\mathfrak{f}$ is a continuous operator.

M. Jerison (Lafayette, Ind.)

CALCULUS OF VARIATIONS

See also 7277.

7459a:

Serrin, James. On a fundamental theorem of the calculus of variations. Acta Math. **102** (1959), 1–22.

7459b:

Serrin, James. A new definition of the integral for non-parametric problems in the calculus of variations. Acta Math. **102** (1959), 23–32.

The author is concerned with the statement assuring that an extremal $u_0(x, y)$, taking the preassigned values on the boundary, gives the minimum for a double nonparametric integral of the calculus of variations

$$I[u] = (R)\int F(x, y, u, p, q)dxdy$$

in a large class of admissible functions $u(x, y)$, if $u_0(x, y)$ can be imbedded in a field of extremals. This theorem has

been proved so far under smoothness conditions at the boundary which the author rightly judges as unrealistic and proves eventually to be superfluous. At the same time the usually required conditions of smoothness for the admissible functions $u(x, y)$ in $I[u]$ are also reduced (for non-negative regular integrands) by defining $I[u]$ by a limit process whose model is taken from surface area theory. Here R denotes an open bounded region of the xy-plane, and F a continuous function of x, y, u, p, q for $(x, y) \in R$, u, p, q real, $F(x, y, u, p, q) \geqq 0$, satisfying the regularity condition $E(x, y, u, p, q, P, Q) \geqq 0$, where E is the usual Weierstrass function. The author denotes by A the class of admissible functions u satisfying: (1) $u(x, y)$ is continuous on $\bar{R} = R \cup R^*$ and takes on the boundary R^* of R preassigned continuous values; (2) u is continuously differentiable in R; (3) $I[u] < \infty$. He denotes by A^* the larger class of admissible functions u satisfying (1) and (3) only. In the first case, $I[u] = I[u, R]$ is a generalized Riemann integral (and an L-integral as well); in the second case (i.e., for u satisfying conditions (1) and (3) only), $I[u] = I_C[u]$ is defined by

$$I_C[u] = \text{Inf} \lim \inf I[v_n, R_n],$$

where $[v_n]$ is a sequence of functions v_n, each v_n being continuously differentiable in a region R_n, with $\bar{R}_n \subset R$, $R_n \to R$, $\max |u - v_n| \to 0$ as $n \to \infty$, and lim inf is taken as $n \to \infty$ and Inf for all sequences $[v_n]$ as above. For $F = (1 + p^2 + q^2)^{1/2}$, $L_C[u]$ is the Lebesgue area of the surface $u = u(x, y)$, $(x, y) \in R$. Also, $I_C[u]$ can be thought of as a generalized Lebesgue area for non-negative regular integrands F, in harmony with previous work on parametric surfaces (McShane, for non-negative regular parametric integrands; Cesari, for all continuous parametric integrands). For non-negative regular integrands $F(x, y, p, q)$ independent of u, the author proves: (I) If the class A, or A^*, is not empty, and $u(x, y)$ is an extremal satisfying condition (1), then $u \in A$, or $\in A^*$, and u gives the minimum of $I[u]$ in A, or A^*. This statement extends to $I[u]$ a known theorem for the Dirichlet integral. The same statement is also extended to a class of integrands F depending on u. For "tame" integrands $F(x, y, p, q)$ (i.e., for non-negative F of class C^2 with $E > 0$ for $(p, q) \neq (P, Q)$, and such that the corresponding Dirichlet problem for the Euler equation is solvable for each circle of some covering of R) the author proves: (II) If $u(x, y)$ gives the minimum of $I[u]$ in the class A, or A^*, then u is of class C^2 in R.—By a field V of extremals is meant here a family of extremals (surfaces) each of the form $u = u(x, y)$, $(x, y) \in R$, covering a region V of the xyz-space, such that (a) through each point of V there passes one and only one surface of the family, (b) the slope functions $p(x, y, z)$, $q(x, y, z)$ defined by the family in V are continuous in V, (c) the intersection of each straight line $x = x_0$, $y = y_0$ with V is a connected set, and (d) any two surfaces of the family have a positive distance. Thus, by this definition of field, no smoothness condition is required at the boundary of R. For non-negative regular integrands $F(x, y, u, p, q)$ the author proves the conclusive result: (III) If an extremal $u(x, y)$, $(x, y) \in R$, satisfies conditions (1) and (3), and is an element of a field V of extremals over R, then u gives the minimum of $I[u]$ in the class A^*.—Concerning the generalized integral $I_C[u]$, the consistency question naturally arises as to whether $I_C[u] = I[u]$ for functions $u(x, y)$ which, besides being continuous in $\bar{R}$, are continuously differentiable in R. This question, which is parallel to an analogous question for Lebesgue area, is answered in the affirmative in the second paper for non-negative regular integrands $F(x, y, u, p, q)$ satisfying either hypothesis: (H1) F has continuous partial derivatives F_{pp}, F_{pq}, F_{qq}, F_{xp}, F_{yq}; (H2) $E > 0$ for $(p, q) \neq (P, Q)$. The proof is actually a verification of the intrinsic property of lower semicontinuity of $I[u]$ for non-negative regular integrands (Tonelli). The following further generalization of the concept of integral $I[u]$ is discussed in the second paper. If $u(x, y)$, $(x, y) \in R$, is supposed to be only measurable in R, then an integral $I_M[u]$ can be defined by the same process used for $I_C[u]$ provided uniform convergence is replaced by almost everywhere convergence, in harmony with analogous generalization of the concept of Lebesgue area (Cesari and Goffman). An analogous consistency proof is given concerning $I_M[u]$ and $I_C[u]$.

L. Cesari (Lafayette, Ind.)

7460:

Lombardi, Lionello. Sull'esistenza del minimo degli integrali di Fubini-Tonelli. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 446–458.

L'Autore, facendo seguito a un precedente lavoro [Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 129–153; MR **21** #831] sulle condizioni sufficienti per la semicontinuità degli integrali di Fubini-Tonelli

$$I(y_1, y_2) = \int_a^b \int_c^d f(x, z, y_1(x), y_2(z), y_1'(z), y_2'(z)) dx dz,$$

ottiene ora teoremi di esistenza del minimo assoluto per $I(y_1, y_2)$.

Appoggiandosi sui concetti, veramente appropriati, di integrale quasi-regolare positivo (q.r.p.) e asintoticamente bilinearizzabile (a.b.), introdotti nel lavoro sopracitato, e applicando il procedimento variazionale di L. Tonelli per gli integrali semplici del calcolo delle variazioni, l'A. dimostra il seguente teorema: Se $I(y_1, y_2)$ è q.r.p. e a.b. e se inoltre la funzione $f(x, z, y_1, y_2, y_1', y_2')$ è inferiormente limitata per (x, z, y_1, y_2) variabile in un dominio B limitato e chiuso e y_1' e y_2' qualunque, ed esistono inoltre due numeri Y e $\alpha > 0$ tale che per $(x, z, y_1, y_2) \in B$ e y_1' e y_2' soddisfacenti alla $|y_1' \cdot y_2'| > Y$, sia

$$f(x, z, y_1, y_2, y_1', y_2') > |y_1'|^{1+\alpha} |y_2'|^{1+\alpha},$$

allora esiste il minimo assoluto di $I(y_1, y_2)$ in ogni classe regolare e completa di bilinee $[y_1(x), y_2(z)]$ ordinarie (coppie di funzioni assolutamente continue per le quali esista finito $I(y_1, y_2)$) contenuta in B. Per classe regolare di bilinee si intende un insieme di bilinee ordinarie tale che ogni successione $\{y_{1,n}(x), y_{2,n}(z)\}$ da esso estratta, verificante la condizione che ad ogni $\varepsilon > 0$ corrisponde un $\delta > 0$ tale che

$$\int_{E_1} |y_{1,n}'(x)| dx \cdot \int_{E_2} |y_{2,n}'(z)| dz < \varepsilon$$

per ogni n e per ogni coppia di insiemi E_1 e E_2 di misura $< \delta$, verifica anche la condizione che ad ogni $\varepsilon > 0$ corrisponde un $\delta > 0$ tale che

$$\int_{E_1} |y_{1,n}'(x)| dx < \varepsilon, \quad \int_{E_2} |y_{2,n}'(z)| dz < \varepsilon$$

per ogni n e per ogni coppia di insiemi E_1 e E_2 di misura $< \delta$. La classe si dice poi completa se contiene ogni bilinea

ordinaria di accumulazione. Vengono dati criteri di regolarità per le classi di bilince e il teorema viene esteso anche al caso di domini illimitati. *E. Magenes* (Pavia)

7461:

de Vito, Luciano. Su un problema di Mayer per gli integrali multipli. Ricerche Mat. 8 (1959), 3–23.

The author extends an existence theorem due to Ciliberto [Ricerche Mat. 6 (1957), 205–236; MR **20** #3468] to the case of r-tuple integrals. A notable difference is the use of the methods of C. B. Morrey, involving the weak topology, in obtaining the theorems on semi-continuity and the final existence theorems. The functional to be minimized is the value (at an extreme corner Q of an r-dimensional interval R in x-space) of the solution of a differential equation $u'=f(x, z(x), p(x), u)$, where $p(x)$ is the gradient matrix of the vector function $z(x)$ and u' denotes the Vitali derivative (mixed derivative or order r) of u. The values of u on the faces of R opposite to Q are prescribed.

L. M. Graves (Chicago, Ill.)

GEOMETRY

See also 7221, 7479, 7488.

7462:

★Szász, Paul. Direct introduction of Weierstrass homogeneous coordinates in the hyperbolic plane, on the basis of the endcalculus of Hilbert. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 97–113. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

A more detailed version of this paper has appeared in Acta Math. Acad. Sci. Hungar. **9** (1958), 1–28 [MR **20** #4223]. *E. Mendelson* (New York, N.Y.)

7463:

★Szmielew, Wanda. Some metamathematical problems concerning elementary hyperbolic geometry. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 30–52. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

Elementary n-dimensional hyperbolic geometry $\mathscr{H}_n$ is based upon Tarski's axioms for elementary Euclidean geometry [cf. Tarski, pp. 16–29 of this volume; MR **21** #4919] except that Euclid's axiom (if u is between x and t and also between y and z, and $x \neq u$, then there exist w and v such that z is between x and v, and y is between x and w, and t is between v and w) is replaced by its negation. First, by an extension of some work by Szász [see review #7462 above], it is indicated that the models of $\mathscr{H}_2$ are precisely certain structures called Hilbert-Szász spaces over real closed fields. Then, by a new method, the hyperbolic calculus of free segments, it is proved that the models of $\mathscr{H}_n$ ($n \geq 2$) are exactly the Klein spaces $\mathscr{K}_n(\mathscr{F})$ over real closed fields $\mathscr{F}$. (Let $\mathscr{F}=\langle F, +, \cdot, \leqq \rangle$ be an ordered field. Let $A_\mathscr{F}$ be $\{\langle x_1, \cdots, x_n\rangle \in F^n \mid \sum_{i=1}^n x_i^2 < 1\}$. If $x, y \in A_\mathscr{F}$, let $x \cdot y = \sum_{i=1}^n x_i \cdot y_i$, and

$$\psi(x, y) = \frac{(1-x\cdot x)\cdot(1-y\cdot y)}{(1-x\cdot y)^2}.$$

Let the betweenness relation $B_\mathscr{F}$ hold for x, y, z in $A_\mathscr{F}$ whenever

$$\psi(x, z) = \frac{\psi(x, y)\cdot\psi(y, z)}{(1+(1-\psi(x, y))^{1/2}\cdot(1-\psi(y, z))^{1/2})^2}.$$

Let the equidistance relation $D_\mathscr{F}$ hold for x, y, z, u in $A_\mathscr{F}$ whenever $\psi(x, y)=\psi(z, u)$. Then $\langle A_\mathscr{F}, B_\mathscr{F}, D_\mathscr{F}\rangle = \mathscr{K}_n(\mathscr{F})$ is called the n-dimensional Klein space over $\mathscr{F}$.) It follows that $\mathscr{H}_n$ ($n \geq 2$) is complete and decidable, but not finitely axiomatizable. Some corollaries are stated, relating to absolute geometry, the common part of Euclidean and hyperbolic geometry. Finally, if $\overline{\mathscr{H}}_n$ is non-elementary n-dimensional hyperbolic geometry, obtained from $\mathscr{H}_n$ by taking the unrestricted non-elementary form of the continuity axiom, then $\overline{\mathscr{H}}_n$ is categorical, and all its models are isomorphic with the Klein space $\mathscr{K}_n(\mathscr{R})$ over the reals. *E. Mendelson* (New York, N.Y.)

7464:

Szmielew, W. Absolute calculus of segments and its metamathematical implications. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 213–220. (Russian summary, unbound insert)

This paper is a summary of results without proofs. The topic concerns elementary geometry in the sense of Tarski [see *The axiomatic method*, pp. 16–29, North-Holland, Amsterdam, 1959; MR **21** #4919; the author's paper in that volume, #7463, should also be consulted]. In particular, absolute geometry is treated for any given dimension greater than 1. Absolute geometry consists of those theorems common to both the Euclidean and hyperbolic theories. However, without distinguishing between the two cases the author builds up the theory of proportion (except, of course, for the results depending on the existence of the fourth proportional which fails in hyperbolic geometry) in a new way which thereby leads to the absolute calculus (or algebra) of (equivalence classes of free) segments. From this point it is then a direct (though somewhat lengthy) step to the representation (or coordinatization) theorems giving either the Cartesian or Klein model when the two cases are finally separated. One important reason for stressing the 'elementary' formulation of the definitions and proofs is that the Archimedean axiom is nowhere applied in the development of the theory of proportion. As a consequence, one finds that the discussion concerns geometries whose underlying field is an arbitrary real-closed field and not simply a subfield of the real numbers.

Dana Scott (Chicago, Ill.)

7465:

Ahrens, Joachim. Begründung der absoluten Geometrie des Raumes aus dem Spiegelungsbegriff. Math. Z. **71** (1959), 154–185.

In seinem Buch *Aufbau der Geometrie aus dem Spiegelungsbegriff* [Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959; MR **21** #6557] hat Ref. die ebene absolute Geometrie in allgemeiner Form entwickelt; insbesondere wurde weder Anordbarkeit noch freie Beweglichkeit gefordert. Verf. gibt eine Begründung der (dreidimensionalen) räumlichen absoluten Geometrie von entsprechender Allgemeinheit.

Verf. formuliert sein Axiomensystem gruppentheoretisch und denkt sich eine Gruppe $\mathfrak{G}$ und ein (gegenüber inneren Automorphismen von $\mathfrak{G}$) invariantes, aus involutorischen Elementen $\alpha, \beta, \cdots$ bestehendes Erzeugendensystem $\mathfrak{S}$ von $\mathfrak{G}$ gegeben ("Grundannahme"). Wenn zwei involutorische Elemente x, y aus $\mathfrak{G}$ ein involutorisches Produkt haben, wird dies durch $x|y$ ausgedrückt ("Strichrelation"). Involutorische Elemente aus $\mathfrak{G}$, die als Produkt von drei paarweise in Strichrelation stehenden Elementen aus $\mathfrak{S}$ darstellbar sind, werden mit $P, Q, R, \cdots$ bezeichnet. Die sechs Axiome lauten: (1) Zu P, Q, R gibt es stets ein α mit $P, Q, R|\alpha$. (2) Aus $P \neq Q$ und $\alpha \neq \beta$ und $P, Q|\alpha, \beta, \gamma$ und $R|\alpha, \beta$ folgt $R|\gamma$. (3) Aus $P \neq Q$ und $\alpha, \beta, \gamma|P, Q$ folgt $\alpha\beta\gamma \in \mathfrak{S}$. (4) Aus $\pi|\vartheta$ und $\alpha, \beta, \gamma|\pi, \vartheta$ folgt $\alpha\beta\gamma \in \mathfrak{S}$. (5) Gilt $P|\alpha, \beta$, so gibt es ein Q mit $PQ \neq QP$ und $Q|\alpha, \beta$. (6) Zu jedem α gibt es ein P mit $P|\alpha$.

Jedem Paar $\mathfrak{G}, \mathfrak{S}$, welches den Axiomen genügt, wird ein Raum $\mathfrak{R}(\mathfrak{G}, \mathfrak{S})$ zugeordnet, dessen Ebenen den $\alpha, \beta, \cdots$ und dessen Punkte den $P, Q, \cdots$ entsprechen und in dem die Orthogonalität von Ebenen durch $\alpha|\beta$ und die Inzidenz durch $P|\alpha$ definiert wird. Die durch innere Automorphismen von $\mathfrak{G}$ induzierten Abbildungen von $\mathfrak{R}(\mathfrak{G}, \mathfrak{S})$ werden Bewegungen des Raumes genannt. Speziell ergibt das Transformieren mit einem Element aus $\mathfrak{S}$ eine Ebenenspiegelung; die Axiome 3 und 4 liefern zwei Formen des Satzes von den drei Ebenenspiegelungen. Die Geometrie in den Ebenen eines Raumes $\mathfrak{R}(\mathfrak{G}, \mathfrak{S})$ erweist sich als ebene absolute Geometrie im Sinne des anfangs genannten Buches.

Ein Raum $\mathfrak{R}(\mathfrak{G}, \mathfrak{S})$ wird elliptisch genannt, wenn er vier paarweise orthogonale Ebenen (ein "Polartetraeder") enthält; die elliptischen Räume sind projektive Räume. Zu jedem nicht-elliptischen Raum $\mathfrak{R}(\mathfrak{G}, \mathfrak{S})$ wird ein projektiver Erweiterungsraum $\mathfrak{R}^*(\mathfrak{G}, \mathfrak{S})$ konstruiert. Die Konstruktion basiert auf den Relationen "α, β, γ liegen im Büschel" und "$\alpha, \beta, \gamma, \delta$ liegen im Bündel", die dadurch definiert werden, dass die Produkte $\alpha\beta\gamma$ bzw. $\alpha\beta\gamma\delta$ verkürzbar sind. Mit diesen Begriffen wird der Anschluss an eine Arbeit von Winternitz [Ann. of Math. (2) **41** (1940), 365–390; MR **1**, 260] hergestellt, in der unter Voraussetzung der Hilbertschen Verknüpfungsaxiome, der Vierkanthypothese und des Satzes von Desargues im Bündel ein projektiver Raum konstruiert wird. In $\mathfrak{R}^*(\mathfrak{G}, \mathfrak{S})$ gelten überdies der Satz von Pappus-Pascal und das Fano-Axiom (im elliptischen Fall sei $\mathfrak{R}^*(\mathfrak{G}, \mathfrak{S}) = \mathfrak{R}(\mathfrak{G}, \mathfrak{S})$). Bei nichteuklidischer Metrik ("ordinärer Fall") wird in $\mathfrak{R}^*(\mathfrak{G}, \mathfrak{S})$, bei euklidischer Metrik ("singulärer Fall") in einer ausgezeichneten Idealebene von $\mathfrak{R}^*(\mathfrak{G}, \mathfrak{S})$, eine projektive Polarität konstruiert; die durch diese "absolute" Polarität in $\mathfrak{R}^*(\mathfrak{G}, \mathfrak{S})$ erklärte Orthogonalität setzt die Orthogonalität aus $\mathfrak{R}(\mathfrak{G}, \mathfrak{S})$ fort.

Das Ergebnis ist: Jeder Raum $\mathfrak{R}(\mathfrak{G}, \mathfrak{S})$ ist Teil eines projektiv-metrischen Raumes, und die axiomatisch gegebene Gruppe $\mathfrak{G}$ ist einer Untergruppe der Bewegungsgruppe eines projektiv-metrischen Raumes isomorph, wobei den Elementen aus $\mathfrak{S}$ Ebenenspiegelungen des projektiv-metrischen Raumes entsprechen. Damit ist der Anschluss an die Cayley-Kleinsche Auffassung der metrischen Geometrie erreicht; es folgt, dass die axiomatisch gegebene räumliche absolute Geometrie im Rahmen der vierdimensionalen metrischen Vektorräume über kommutativen Körpern von Charakteristik $\neq 2$ algebraisierbar ist.

F. Bachmann (Kiel)

7466:

Junkers, Wilhelm. Über Geometrien mit einem unsymmetrischen Orthogonalitätsbegriff. I. Eine Begründung der anisotropen ebenen quasielliptischen Geometrie. Arch. Math. **10** (1959), 206–215.

The author continues his axiomatic study of geometries with an unsymmetric orthogonality relation and in particular those which he refers to as anisotropic plane quasi-elliptic geometries (definitions and axioms below). One of the principal results proclaims that each geometry satisfying his axioms is either pole-free or quasielliptic.

There is an ordered binary relation $\perp_>$ in the class of lines; $g \perp_> h \equiv h <\!\perp g$. If $g <\!\perp g$, g is called isotropic; otherwise g is anisotropic. If no lines are isotropic the geometry is anisotropic. Axioms of incidence: 1.1. Two points determine a line. 1.2. There exist three points not on a line. Axioms of orthogonality: 2.1+. To each line g and each point P there is at least one line h passing through P with $h \perp_> g$. 2.1−. Similarly, with $<\!\perp$. 2.2+. If through a point P there are two distinct lines h_1 and h_2 with $h_1 \perp_> g$ and $h_2 \perp_> g$ then all lines h through P satisfy $h \perp_> g$. 2.2−. Similarly, with $<\!\perp$. 2.3. There are no isotropic lines. 2.4. If $h \perp_> g$ then h and g intersect. Definitions: P is a + pole of line g if all lines through P are perpendicular to g, i.e., $h \perp_> g$ for all h through P. P is a − pole of line g if $h <\!\perp g$ for all h through P. Dual definitions hold for + and − polars. A geometry satisfying the axioms which is free of + poles or − poles is called pole-free, and is quasi-elliptic if each line has exactly one + pole and one − pole and dually each point has exactly one + polar and one − polar.

In addition to the principal theorem mentioned in the first paragraph examples of anisotropic plane quasi-elliptic geometries are produced from among the class of matrices over a field, with appropriate definitions, and the possibility of introducing an orthogonality relation into a projective geometry so that the resulting space is an anisotropic quasielliptic plane is considered.

L. M. Kelly (East Lansing, Mich.)

7467:

Sydler, J.-P. Sur quelques polyèdres équivalents obtenus par un procédé en chaînes. Elem. Math. **14** (1959), 100–109.

Two polyhedra are equivalent if each can be decomposed into smaller polyhedra which are congruent in pairs. Dehn established necessary conditions for two polyhedra to be equivalent. In this paper the author establishes sufficient conditions. The method involves the study of chains of planes $\pi_1, \pi_2, \cdots, \pi_n$ (all passing through a point) with π_i orthogonal to π_{i+1} $(i = 1, \cdots, n-1)$, and trigonometric relations among the angles between the traces of the "links" of the chain in another plane.

W. Moser (Winnipeg, Man.)

7468:

Eves, Howard. Philo's line. Scripta Math. **24** (1959), 141–148.

L'auteur se propose de réunir dans un seul article certaines des propriétés les plus connues de la droite de Philon (droite passant par un point donné situé à l'intérieur d'un angle donné telle que les côtés de l'angle interceptent sur elle un segment de longueur minimum), de montrer le rôle joué par cette droite dans la duplication du cube (d'où l'appellation de la droite de Philon) et l'impossibilité de la construire en général par l'emploi de la règle et du compas, et d'esquisser quelques généralisations. *F. Şemin* (Istanbul)

7469:

Deaux, R. Équation cubique et triangle. Mathesis **68** (1959), 37–42.

Let $T=ABC$ be the triangle in the Gaussian plane determined by the zeros a, b, c of the cubic polynomial $P(z)=a_0z^3+3a_1z^2+3a_2z+a_3$, G the center of gravity of T and W_1, W_2 its isodynamic points. The author shows a few properties of T: (1) If $r=|GW_1/GW_2|$, where GW_1 and GW_2 are distances from the center of gravity to the isodynamic points, then $|AW_1/AW_2|=|BW_1/BW_2|=|CW_1/CW_2|=r^{1/3}$; (2) $\sphericalangle W_1GW_2=3\sphericalangle W_1AW_2=3\sphericalangle W_1BW_2=3\sphericalangle W_1CW_2 \pmod{2\pi}$; (3) the center of the circle (O) circumscribed about T and the radius of (O), the point of Lemoine and the point of Feuerbach are found in terms of the coefficients of $P(z)$. There exist infinitely many triangles with the same isodynamic points (and the same points of Lemoine, Beltrami and Brocard) inscribed in (O). The author discusses the locus of the centers of gravity of the inscribed triangles and the locus of the foci of the Steiner's ellipse inscribed into them. The loci are some circles determined by W_1, W_2, and r. *J. W. Andrushkiw* (Newark, N.J.)

7470a:

Tschupik, Josef P. Über die Abbildung des projektiven R_n durch zwei Projektionen. Monatsh. Math. **63** (1959), 1–18.

7470b:

Tschupik, Josef P. Über lineare Zweibildersysteme im projektiven R_n. Monatsh. Math. **63** (1959), 214–227.

Im Anschluß an Hohenberg [Monatsh. Math. **61** (1957), 54–66; MR **18**, 921] untersucht der Verfasser weitgehende Verallgemeinerungen des bekannten Zweibilder-Prinzips der darstellenden Geometrie. Es werden dabei 2 Hohenbergsche P-Projektionen mit den Zentren Z_i je auf die Bildräume Π_i gemeinsam betrachtet. Z_i und Π_i sollen dabei je die Dimensionen $n-r_i$ und r_i-1 haben. Verf. nennt zunächst a-Systeme solche Paare von Projektionen, bei denen die Räume $Z=Z_1\cap Z_2$, $X=\Pi_1\cap\Pi_2$ und $X\cap(Z_1\cup Z_2)$ minimale Dimensionen haben. Diese a-Systeme sind im einzelnen danach zu unterscheiden, ob $r_1+r_2<n+1$, $=n+1$, $=n+2$ oder $>n+2$ ist. Die kotierte Projektion und das übliche Zweibilder-Prinzip sind Sonderfälle davon. Weiterhin werden eingeführt: z-Systeme, bei denen die Dimension z von Z den Minimalwert $n-(r_1+r_2)$ überschreitet und $\geqq 0$ ist; x-Systeme, bei denen die Dimension x von X ihren Minimalwert überschreitet. xz-Systeme sind solche, bei denen beide Besonderheiten gleichzeitig auftreten. In allen Fällen wird im einzelnen untersucht, ob und mit welcher Unbestimmtheit man von Bildpunktpaaren auf Urpunkte schließen kann.

In der zweiten Arbeit, einer Fortsetzung der ersten, untersucht der Verf. dann die sog. y-Systeme, bei denen der Raum $O=Z_1\cup Z_2$ zu X spezielle Lage besitzt. Zwischen allen genannten Typen gibt es wiederum Kombinationsmöglichkeiten, deren Existenz im einzelnen untersucht wird. Gewiße gemeinsame Eigenschaften aller Abbildungsprinzipien werden am Schluß der zweiten Arbeit zusammengestellt. *W. Burau* (Hamburg)

7471:

Tschupik, Josef P. Schrägrisse als Hilfsmittel zur Konstruktion von Durchdringungskurven. Elem. Math. **14** (1959), 78–82.

In this paper the author gives some practical hints for the construction of the curve of intersection of two given surfaces with methods of descriptive geometry.

M. Piazzolla-Beloch (Ferrara)

7472:

Vala, Klaus. A propos du théorème de Pohlke. Ann. Acad. Sci. Fenn. Ser. A. I. no. 270 (1959), 5 pp.

Dans cette note l'auteur donne une proposition générale pour la décomposition d'une transformation linéaire d'un espace unitaire, de laquelle le théorème de Pohlke s'obtient comme cas particulier.

M. Piazzolla-Beloch (Ferrara)

7473:

Deaux, R.; et Depunt, J. Quadruples hyperboloïdiques. Mathesis **68** (1959), 7–13.

Dans la présente note l'A. donne des critères de dégénérescence des quadruples hyperboloïdiques liés à des quadriques réelles tangentes aux faces ou à trois arêtes concourantes d'un tétraèdre réel.

M. Piazzolla-Beloch (Ferrara)

7474:

Zappa, Guido. Piani affini finiti con traslazioni. Ricerche Mat. **7** (1958), 241–253.

This paper introduces the concept of a partial affine plane. For this it is required that two lines intersect in at most one point, that given a line L and a point P not on L there is a unique line L' through P parallel to L. Parallel lines may not intersect and it is required that the parallel property be symmetric and transitive. A partial affine plane is called regular if there is one class of parallel lines (the principal class) with the property that every other line intersects every line of this parallel set. The principal class will consist of m lines each containing n points and every other class of parallels will consist of n lines each containing m points. Some properties of finite affine planes are shown to hold for regular partial affine planes. Thus if there are non-identical translations in more than one direction, the translations form an Abelian group and every element $\neq 1$ of the group is of infinite order or of the same prime order p. A partial affine plane is a partial translation plane if the translations are transitive on its points. It is proved that a partial translation plane

containing a translation of order greater than 2 contains a homothetic transformation of order 2 for an arbitrarily assigned center.

The translations and homothetic transformations of the Hughes plane of order 9 are determined. It is shown that an affine finite plane of order 9 is either a translation plane or has at least 9 sets of transitivity under its translation group. *Marshall Hall, Jr.* (Pasadena, Calif.)

7475:

Edge, W. L. Quadrics over GF(2) and their relevance for the cubic surface group. Canad. J. Math. **11** (1959), 625–645.

For $n=2, 3, 4, 5$ the author classifies quadrics Q in projective spaces $[n]=\mathrm{PG}(n, 2)$ over a field of two marks 0, 1, and also classifies linear subspaces of $[n]$ by their relation to Q. Tables of incidences between such subspaces when $n=4$ or 5 are related to those in C. M. Hamill's paper [Proc. Lond. Math. Soc. (3) **3** (1953), 54–79; MR **14**, 1060]. A quadric Q in [4] consists of the 15 points c_{ij} of intersection of the quadric $\mathscr{L}$ in [5] defined by $\sum x_i x_j=0$, $0\leqq i<j\leqq 5$, with the hyperplane $\sum_0^5 x_i=0$. The non-ruled quadric $\mathscr{L}$ in [5] consists of 27 points that correspond to the 27 lines on the cubic surface in such a way that three points on one of the 45 generators of $\mathscr{L}$ correspond to the three sides of a triangle on the cubic surface. (In [5] if $a_1, a_2, \cdots, a_6$ should denote the columns of the 6×6 unit matrix I, u their sum, and $b_j=u-a_j$, $c_{ij}=b_j-a_i=b_i-a_j$, then the 27 vertices would be assigned the Schläfli labels a_i, b_j and c_{ij}.) Each of the 36 points p of [5] not on $\mathscr{L}$ is (like u) the center of perspective for two hexads (like the a_i and b_j) that correspond to lines of a double six, and each p determines one of 36 involutory matrices J_p that generate a group G of order 51840 isomorphic to the cubic surface group. (If $\tilde{p}$ denotes the row vector for the polar of p with respect to J_p, then $J_p=I+p\tilde{p}$.) Matrices J_{p_1} and J_{p_2} commute or not according as p_1p_2 is tangent or skew to $\mathscr{L}$. The subgroup S of 6! permutation matrices contains representatives of 11 of the 25 classes of G, one for each partition of 6. (For $p=a_1+a_2+a_3$ the involution J_p serves with S to generate G.) The "bitangent" group Γ in which G is a subgroup of index 28 is found in a final section to be isomorphic to the 6×6 symplectic group of matrices μ for which $\mu' S\mu=S$, where S is a non-singular skew matrix, which may be taken to be the matrix of $\mathscr{L}$. Under Γ a polarity rather than a quadric is left invariant. Cosets of G in Γ correspond to the 28 non-ruled quadrics among the 64 quadrics having the same polarity.

J. S. Frame (East Lansing, Mich.)

7476:

Coxeter, H. S. M. Polytopes over GF(2) and their relevance for the cubic surface group. Canad. J. Math. **11** (1959), 646–650.

Referring to Edge's immediately preceding paper [#7475 above], the author identifies his non-ruled quadric in PG(5, 2) with a modular counterpart of Gosset's 27-vertex semi-regular polytope 2_{21}, and notes that the remaining 36 points in PG(5, 2) form an elliptic 1_{22} obtained by identifying opposite vertices of Elte's uniform polytope 1_{22}. The quadratic form

$$E_6=\sum (x^i)^2-x^1x^2-x^2x^3-x^3x^4-x^4x^5-x^5x^6$$

includes for $E_6=0$, the following 27 points of PG(5, 2): six points $a_1=(0\,1\,0\,0\,0;\,1)$, $a_2=(1\,1\,0\,0\,0;\,1)$, $a_3=(1\,0\,0\,0\,0;\,1)$, $a_4=(1\,0\,1\,0\,0;\,1)$, $a_5=(1\,0\,1\,1\,0;\,1)$, $a_6=(1\,0\,1\,1\,1;\,1)$ with sum $u=(1\,0\,1\,0\,1;\,0)$; six points $b_j=u+a_j$; and 15 points $c_{ij}=u+a_i+a_j$. Descriptions already published for the subspaces of 2_{21} and 1_{22} are related to the subspaces described in Edge's paper.

J. S. Frame (East Lansing, Mich.)

7477:

★Tits, J. Sur les analogues algébriques des groupes semi-simples complexes. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 261–289. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

Pour obtenir des analogues des groupes de Lie simples sur des corps quelconques [voir C. Chevalley, Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457], l'auteur définit des géométries associées aux schémas. Une géométrie consiste d'un ensemble E réparti en familles F_i, d'une relation d'incidence et d'un groupe A d'automorphismes conservant chaque famille et l'incidence. Géométrie résiduelle d'une telle géométrie par rapport à un élément $a\in F_i$ est l'ensemble des éléments incidents à a et n'appartenant pas à F_i, où la répartition en familles, la relation d'incidence et le groupe s'obtiennent par restriction. (Par exemple, pour la géométrie projective à n dimensions la géométrie résiduelle d'un élément à k dimensions est la somme directe des géométries à k et à $n-k$ dimensions dans a et autour de a.) Un système de géométries est appelé complet, si toute géométrie résiduelle d'une géométrie du système est isomorphe à une géométrie du système. Les géométries d'un système complet peuvent être caractérisées par des schémas (où la géométrie résiduelle d'un élément a s'obtient en retirant le nœud correspondant à la famille de a), si et seulement si les géométries résiduelles de deux éléments de la même famille sont isomorphes.

Si G est un groupe où on a marqué des sous-groupes $G_1, \cdots, G_n$, on définit la famille F_i comme l'ensemble des classes à droite de G_i dans G, deux éléments de $E=\bigcup F_i$ étant incidents si leur intersection n'est pas vide; A est défini par G opérant à droite sur les éléments de E, donc A isomorphe à $G/\bigcap g^{-1}G_ig$. On peut formuler des conditions nécessaires et suffisantes, sous lesquelles une géométrie est de cette espèce spéciale. Par exemple, pour que toutes les géométries d'un système complet soient de cette espèce: A est transitif sur chaque famille, aucune famille n'est vide, deux éléments d'une même famille ne sont pas incidents.

Pour adjoindre des géométries aux schémas de groupe semi-simple, on se donne les géométries à deux dimensions (schémas de deux nœuds à liaison nulle, simple, double et triple) et on postule axiomatiquement quelques propriétés de géométries en général. Partant pour un corps quelconque de définitions généralisant les définitions classiques, on trouve les groupes de Chevalley, étudiés spécialement au cas d'un corps fini. L'auteur admet même le corps de "caractéristique 1", où l'on part des géométries du polygone à 2, 3, 4, 6 côtés, pour obtenir comme groupes associés aux schémas supérieurs ceux de Weyl.

H. Freudenthal (Utrecht)

CONVEX SETS AND GEOMETRIC INEQUALITIES

7478:

Handscomb, D. C. A note on a problem of Rényi. Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 87–88. (Hungarian and Russian summaries)

Consider the $n \times n$ matrices $X = \{x_{ij}\}$ which, subject to the conditions $\{x_{ij} \geq 0\}$, $\{\sum_j x_{ij} = g_i\}$, and $\{\sum_i x_{ij} = h_j\}$, minimize the form $\sum_{ij} c_{ij} x_{ij}$. A certain number of matrix elements will be zero. If more than one matrix satisfies the conditions select the one with the largest number of zero elements. As $\{g_i\}$, $\{h_j\}$, and $\{c_{ij}\}$ vary the matrices X will have various numbers of zero elements. Then $f(n)$, the smallest such number for a given n, is $(n-1)^2$.

S. Sherman (Philadelphia, Pa.)

7479:

Davis, Chandler. The set of non-linearity of a convex piecewise-linear function. Scripta Math. **24** (1959), 219–228.

The author considers a partition of the whole affine n-space into a finite number of convex n-cells in the manner of a topological n-complex. (Each finite n-cell is a polytope, and each infinite n-cell likewise has flat boundaries.) The honeycomb is said to be "simple" if the number of cells at each vertex is exactly $n+1$. He seeks conditions for the existence of a non-trivial convex function f (of any given system of affine coordinates) which is linear throughout each cell and continuous everywhere. The honeycomb is said to be "realizable" if such a function f exists, "uniquely realizable" if any other such function is of the form $\alpha f + g$ where α is real and g is linear throughout the whole space. He finds that a one-dimensional honeycomb (consisting of two rays with or without intervening segments) is always realizable, but if there are any segments the realization is not unique. On the other hand, every simple honeycomb with $n \geq 3$ is uniquely realizable. There remains the interesting case when $n = 2$, that is, the case of a tessellation. Let $A_1A_2 \cdots A_m$ be one face (or 2-cell) of a simple tessellation (whose vertices and edges form a trivalent graph). The "intrinsic slope" of the third edge from A_i is the ratio in which the line of that edge divides the segment $A_{i-1}A_{i+1}$. The tessellation is found to be realizable if and only if the product of the intrinsic slopes round each finite face is 1, and in this case it is uniquely realizable.

H. S. M. Coxeter (Cedar Falls, Iowa)

7480:

Grünbaum, Branko. Affine-regular polygons inscribed in plane convex sets. Riveon Lematematika **13** (1959), 20–24. (Hebrew summary)

The author shows that in any plane convex set there can be inscribed two different affine-regular pentagons with one side in a given direction, and in any central plane convex set there can be inscribed at least one regular octagon. He also gives a simple proof of the known fact that any plane convex set having the property that any family of mutually intersecting translates has a point common to all members of the family is necessarily a parallelogram.

H. G. Eggleston (London)

7481:

Imre, Margit. On an extremal property of the penta- and hexagram. Mat. Lapok **9** (1958), 289–293. (Hungarian. Russian and English summaries)

A double oval is a closed curve with nowhere negative curvature and total curvature 4π. Its area is defined as $a = \frac{1}{2}\int Or \times dr$. Among double ovals with 5 self-intersections the area of the smallest lunale divided by a is largest for the affine regular pentagram. A similar statement holds for hexagrams.

P. Ungar (New York, N.Y.)

7482:

Sachs, Horst. Über eine Klasse isoperimetrischer Probleme. I, II. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **8** (1958/59), 121–126, 127–134.

The author considers the function $M_\lambda[\mathfrak{C}]$ defined for closed curves $\mathfrak{C}$ (of length l) and $\lambda \geq -1$ by

$$\left\{L^{-2}\int_{\mathfrak{C}}\int_{\mathfrak{C}} ds\,ds'\,|PP'|^\lambda\right\}^{1/\lambda},$$

where P and P' vary independently on $\mathfrak{C}$ and ds, ds' are elements of arc length. For $\lambda = 0$ and $\lambda = \infty$ there are the usual modifications,

$$M_0[\mathfrak{C}] = \exp\left\{L^{-2}\int_{\mathfrak{C}}\int_{\mathfrak{C}} ds\,ds' \log|PP'|\right\},$$

$$M_{+\infty}[\mathfrak{C}] = \operatorname{Max} L^{-1}|PP'|.$$

He shows that (a) for n-dimensional curves a circle gives a maximum for the range $-1 < \lambda \leq 2$ and a double segment a maximum for $\lambda = \infty$; there is no minimum for $\lambda = 2$ or for $\lambda = \infty$; (b) for plane convex sets similar results hold except that there is a minimum when $\lambda = 2$, namely the equilateral triangle, and a minimum when $\lambda = \infty$, namely curves of constant width; (c) if the curve is plane, convex, centrally symmetric, then the same results hold except that the minimum when $\lambda = 2$ is given by a parallelogram and when $\lambda = \infty$ by a circle.

For the range $2 < \lambda < \infty$ no precise results are given, but it is shown that the value of the function when $\lambda = 3$ is less for the double segment than for a circle whereas when $\lambda = 4$ the position is reversed.

H. G. Eggleston (Pinner, Middlesex)

7483:

Sachs, Horst. Über eine Klasse isoperimetrischer Probleme. III. Extremwerte des Entfernungsmittels $K_\lambda[\mathfrak{C}] = \{(4\pi^2)^{-1}\int_{\mathfrak{C}}\int_{\mathfrak{C}} d\vartheta\,d\vartheta'\,|PP'|^\lambda\}^{1/\lambda}$ für geschlossene konvexe Kurven $\mathfrak{C}$ gegebener Länge. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **8** (1958/59), 345–350.

The author considers another mean value of chord length of curves $\mathfrak{C}$ (see preceding review). He considers the function

$$K_\lambda[\mathfrak{C}] = \left\{(4\pi^2)^{-1}\int_{\mathfrak{C}}\int_{\mathfrak{C}} d\vartheta\,d\vartheta'\,|PP'|^\lambda\right\}^{1/\lambda}$$

with the usual modifications when $\lambda = 0$ and $\lambda = \infty$. The author shows that for all closed convex curves of given length, this functional has its maximum value for $1 \leq \lambda \leq \infty$ when the curve is a double segment, and its least value for $2 \leq \lambda < \infty$ when the curve is a circle and for $\lambda = \infty$ when

the curve is a curve of constant width. It is known that the maximum for $\lambda=0$ occurs for a circle. The proof depends upon a consideration of Fourier's series expansion of the support function.

H. G. Eggleston (Pinner, Middlesex)

7484:

Sachs, Horst. Über eine Klasse isoperimetrischer Probleme. IV. Beweis des Satzes, dass unter allen geschlossenen Kurven gegebener Länge die Nadel und nur diese für jedes $\lambda \geq 1$ den grössten Wert für den transfiniten Durchmesser vom Grade λ liefert. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/59), 351–356.

For a point set X, the transfinite diameter of degree λ is defined by

$$D_\lambda[X] = \lim_{\nu\to\infty} \operatorname{Max}\left\{(\nu(\nu-1))^{-1}\sum_{i\neq k}^{1,\cdots,\nu}|P_iP_k|^\lambda\right\}^{1/\lambda} \quad \text{for } \lambda\neq 0,$$

$$= \lim_{\nu\to\infty} \operatorname{Max}\left\{\prod_{i\neq k}^{1,\cdots,\nu}|P_iP_k|\right\}^{1/\nu(\nu-1)} \quad \text{for } \lambda=0,$$

where the maximum is taken over all sets of ν points $P_1, \cdots, P_\nu$ contained in X. The author shows that for a curve $\mathfrak{C}$, it is possible to find a non-decreasing function S of the arc length of $\mathfrak{C}$ with $\int_{\mathfrak{C}} dS = Q > 0$ for which the functional

$$M_\lambda[\mathfrak{C}, S] = \left\{Q^{-2}\int_{\mathfrak{C}}\int_{\mathfrak{C}} dS\,dS'|PP'|^\lambda\right\}^{1/\lambda}$$

is a maximum and that this maximum is equal to the transfinite diameter of degree λ of $\mathfrak{C}$. He uses this to show that of all closed curves of given length in n-dimensional Euclidean space, the double segment has the largest value for the transfinite diameter of degree λ where $\lambda \geq 1$.

H. G. Eggleston (Pinner, Middlesex)

7485:

Sachs, Horst. Zur Theorie gewisser geometrischer Funktionale und zugehöriger isoperimetrischer Probleme. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/59), 357–364.

The author considers a general form of extremal problem. He considers it in an abstract sense and establishes the equivalence of three sets of conditions on the function concerned. He also considers in detail certain cases which can be solved in terms of differential coefficients of Fourier's series expansion of curves which are variations from a circle. In particular he considers problems in which two of the following functions are selected, one is kept constant and the extremal values of the other are sought: area of set, length of perimeter, minimum moment of inertia, outer conformal radius and mean value of chord length.

H. G. Eggleston (Pinner, Middlesex)

7486:

Sachs, Horst. Über Verallgemeinerungen der Steinerschen Symmetrisierung. I. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/59), 365–374.

The author considers systematically various possible generalisations of Steiner symmetrisation of plane or three-dimensional sets. He gives a number of examples and conditions on the symmetrisation processes which will ensure the variation in appropriate senses of certain functions. He also shows that the method which he uses transforms open sets into open sets.

H. G. Eggleston (Pinner, Middlesex)

7487:

Ionin, V. K.; and Suvorov, G. D. On the components of the level sets of the function: distance to a plane continuum. Dokl. Akad. Nauk SSSR **129** (1959), 496–498. (Russian)

Let K be a bounded continuum of diameter d in the euclidean plane, and $e(x, K)$ the distance of a variable point x from K. The paper investigates the level "curves" E_ρ: $e(x, K)=\rho$. The set-theoretical results are much too involved to be stated here, but there are some easily stated interesting metric results analogous to those for convex curves.

Let G be a simply connected domain with boundary E. The accessible boundary points of E have a natural cyclic order. For any finite set of such points $p_{n+1}=p_1, \cdots, p_n$ in this order, form $\sum_{i=1}^n e(p_i, p_{i+1})$ and denote the sup of these sums as the length of E.

Let E_ρ' be a component of E_ρ and G_ρ' that component of the complement of E_ρ' which contains K. If $e'(\rho)$ is the length of E_ρ', then $e'(\rho) \leq \pi(3^{-1/2}d+\pi\rho)^2\rho^{-1}$ if G_ρ' is bounded; otherwise $e'(\rho) < \pi d^2/3\rho$. Denote by Δ the set of points x of G_ρ' with $e(x, E_\rho') < \delta < \rho$. The component $\Gamma'(\rho, \delta)$ of the boundary of Δ disjoint from E_ρ' has length $e'(\rho) \pm 2\pi\delta$ and the area of $\bigcup_{0<\delta<\rho}\Gamma'(\rho, \delta)$ is $\rho e'(\rho) \pm \pi\rho^2$.

H. Busemann (Los Angeles, Calif.)

7488:

Barthel, Woldemar. Zur Minkowski-Geometrie, begründet auf dem Flächeninhaltsbegriff. Monatsh. Math. **63** (1959), 317–343.

The paper partly consists of new results and partly of affinely invariant treatments of results established by the reviewer with the help of an auxiliary euclidean metric. In an n-dimensional affine space W^n let an affine p-area be defined ($1 \leq p \leq n$), i.e., to a parallelepiped spanned by the vectors $x_1, \cdots, x_p$ an area $F(x_1, \cdots, x_p) = F(x_j)$ is assigned with the properties $F(x_j) > 0$ for $X_p = x_1 \wedge \cdots \wedge x_p \neq 0$, $F(\lambda x_j) = \lambda^p F(x_j)$ for $\lambda \geq 0$, $F(x_j)$ depends only on X_p. The p-area is convex if $F(x_1+x_1', x_2, \cdots, x_p) \leq F(x_1, x_2, \cdots, x_p) + F(x_1', x_2, \cdots, x_p)$. Interpreting $F(x_j)$ as a function $f(X_p)$ on the Grassmann cone, convexity means that $f(X_p+Y_p) = f(X_p)+f(Y_p)$ if X_p, Y_p and X_p+Y_p are simple p-vectors. If a volume has been introduced in W^n, then an isomorphism of the $(n-1)$-vectors on the vectors of the dual space can be defined by associating with an $(n-1)$-vector $x_1 \wedge \cdots \wedge x_{n-1}$ a pair of hyperplanes parallel to this vector and such that the vectors $x_1, \cdots, x_{n-1}$ and a (suitably oriented) vector leading from one of these hyperplanes to the other span a parallelepiped of volume 1. A treatment of Minkowski areas follows, i.e., of the affine areas in a Minkowski space for which the unit sphere in a p-flat has the same volume as a euclidean unit p-ball. Call the line y transversal to the p-flat $(x_1, \cdots, x_p)$, if $F_p(x_j+\lambda y_j) \geq F_p(x_j)$. If F_p is convex then for a given $(p+1)$-flat L_{p+1} containing the oriented p-flat L_p, a line transversal to L_p at a point $z \in L_p$ exists. {The theorem is ascribed to the reviewer, but this special case goes back to Carathéodory.

The connection with the isoperimetric problem and transversality to L_p of flats not intersecting L_p in points only seems to be due to the reviewer.} The transversals to L_p at z in L_{p+1} form a convex cone. The intersections of these cones, for fixed z, L_p, variable $L_{p+1} \supset L_p$, with a hyperplane H not through z, form a convex family of convex sets in H. For given, not necessarily convex F_1, F_{n-1}, the relations between indicatrix and isoperimetrix (normalized solution of the isoperimetric problem for the area F_{n-1}) are discussed in an invariant way. The corresponding isoperimetric and isodiametric inequalities are stated. *H. Busemann* (Los Angeles, Calif.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 7268, 7275.

7489:

Kosiński, A. Some theorems about two-dimensional polyhedra. Fund. Math. **47** (1959), 1–28.

The main purpose is to provide a topological characterization of finite 2-dimensional polyhedra which is based entirely on the concepts and methods of set-theoretic topology. Since both the euclidean plane and the graph (=finite 1-dimensional polyhedron) are known to be capable of such a basis, they may be utilized freely. Accordingly, if K is a space, reg K is defined to be the subset of K consisting of those points having neighborhoods homeomorphic to the euclidean plane. Then a finite, 2-dimensional polyhedron may be completely characterized as a compact metric space K which is (1) an ANR, (2) expressible as the union of sets A and B such that $A \subset \text{reg } K$ and B is a graph, and (3) such that almost all points p of B have arbitrarily small neighborhoods U such that for every component C of $U-B$ the set $C \cup (U \cap B-p)$ is connected. Three alternative conditions for condition (3) are obtained [for one of these see A. Kosiński, Bull. Acad. Polon. Sci. Cl. III **2** (1954), 321–323; MR **16**, 158].

Since the notions of graph and plane are only indirectly in the province of set-theoretic topology, another characterization is derived explicitly in set-theoretic terms. It makes use of the concept of "r-polyhedron", which is inductively defined as follows. Let K be an n-dimensional ANR, $n>0$. If $p \in A \subset K$, then p is an r-point of K rel. A if p has arbitrarily small neighborhoods U such that for every $q \in U \cap A$, the boundary of U is a deformation retract of $\overline{U}-q$. An r-point of K rel. K is called simply an r-point of K [A. Kosiński, Fund. Math. **42** (1955), 111–124; MR **17**, 654]. Now let K_1 be the set of r-points of K at which $\dim K=n$; and for $i \geq 2$, let K_i be the set of r-points of K rel. $K-\bigcup_{j<i} K_j$. Let every finite space be an r-polyhedron. Then K is an r-polyhedron if (a) K_1 is open, (b) $K=\bigcup K_i$, (c) $K-\bigcup_{j \leq i} K_j$ is an r-polyhedron of dimension $\leq n-i$. Every finite polyhedron is an r-polyhedron for some n, and the converse is proved to hold for $n \leq 2$. It is not known whether r-polyhedra are always polyhedra; the author feels that this is probably not the case.

By way of justification, the author applies his main result to show that if $A \times B$ is a polyhedron and $\dim A \leq 2$, then A is a polyhedron (see A. Kosiński [Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 325–328; MR **16**, 159]). This provides a solution of a problem of Borsuk (Bing showed that without the condition "$\dim A \leq 2$", the answer is negative [Ann. of Math. (2) **70** (1959), 399–412; MR **21** #5953]). Also, the main result implies the known theorem that in dimension 2 local triangulability is equivalent to triangulability.

Auxiliary results are obtained concerning manifolds with boundaries imbeddable in the plane, compactifications of 2-dimensional open manifolds, and 2-dimensional pseudomanifolds. In particular, it is shown that if K is a 2-dimensional ANR such that $\dim (K-\text{reg } K) \leq 0$, then K is a polyhedron; and accordingly that a 2-dimensional continuum K is a closed pseudomanifold if and only if it is an ANR, reg K is connected and $\dim (K-\text{reg } K) \leq 0$. *R. L. Wilder* (Ann Arbor, Mich.)

7490:

★Boland, Johan Christoph. Bijdrage tot de topologie der vlakke, continue afbeeldingen. [Contribution to the topology of plane continuous mappings.] Thesis, Universiteit van Amsterdam. G. Van Soest, Amsterdam, 1959. 52 pp.

A (separable metric) space X is said to possess the property α_Y if every one-to-one continuous mapping of X into Y is a homeomorphism. The author restricts to the cases: $Y=$ one- or two-sphere, and $X \subset Y$. For the 1-sphere a necessary and sufficient condition reads: X is compact or open in Y; X is homogeneously one-dimensional; the components space of X is compact; all components of $Y \backslash X$ are compact. The case of the 2-sphere Y is more difficult and more interesting. The author supposes X to be arcwise connected. His main idea is very remarkable though not clearly enough explained. For any $p \in X$, $q \in \overline{X} \backslash X$, $q=\lim q_n$, $q_n \in X$, we will try to prevent q_n from being mapped into a neighborhood of p under a one-to-one continuous map of X into Y. So we will postulate some obstruction in X between p and the "ideal" point q of X, by which any connexion between p and q will be blocked. X is called a complete separation of Y if any pair p, q ($p \in X$, $q \in \overline{X} \backslash X$) is separated in Y by some topological circle of X. {This terminology is highly misleading, since a complete separation need not be a separation at all.} It is easy to see that the property of complete separation does not imply α_Y. A more efficient obstruction is needed. This can be a figure homeomorphic to the graph A_1 of edges of a tetrahedron with p and q on opposite edges, or to the graph A_2 of edges of a double triangular pyramid with p and q in the opposite vertices. Indeed any connexion between the images of p and the "ideal" q would imply a spherical imbedding of a Kuratowski graph originating from connecting the images of p and q.

This justifies the following definition: X is called an absolute complete separation of Y if for any $p \in X$ and $q \in \overline{X} \backslash X$ one of the graphs A_1, A_2 can be imbedded into X in such a way that the components of p and q (i.e., near q) in $X \backslash A_i$ have limit points on opposite edges of A_i. The author proves the following conjunction to be equivalent to α_Y: X is an absolute complete separation of Y; if $Y \backslash X$ has an open component, X is connected; if all components of $Y \backslash X$ are closed, X has at most a finite number of components. *H. Freudenthal* (Utrecht)

7491:

Kaufmann, Iosif. Une propriété des domaines fermés. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 133–134 (1959). (Romanian. French and Russian summaries)

Consider the closed domain (=connected set, closure of its interior) R in a Euclidean n-space. This note shows that if R is expressed as the union of two closed subsets, then the cardinal number of the intersection of the closed subsets is equal to the minimum of the cardinal numbers of the sets. *M. M. Day* (Urbana, Ill.)

7492:

Segal, Jack. Hyperspaces of the inverse limit space. Proc. Amer. Math. Soc. **10** (1959), 706–709.

Let X be a metric continuum and $C(X)$ the space of all nonempty subcontinua of X provided with the topology induced by the usual Hausdorff metric. Then $C(X)$ is acyclic in all dimensions (Čech homology). This was known before only for Peano continua, where $C(X)$ is contractible [J. L. Kelley, Trans. Amer. Math. Soc. **52** (1942), 22–36; MR **3**, 315]. The author obtains his result by using the fact that X is the limit of an inverse sequence (X_i, f_i) of Peano continua (actually polyhedra), and by observing that $C(X)$ is the limit of the inverse sequence $(C(X_i), f_i^*)$, with obvious maps $f_i^*: C(X_{i+1}) \to C(X_i)$ induced by $f_i: X_{i+1} \to X_i$. Dimension of $C(X)$ is discussed too. *S. Mardešić* (Zagreb)

7493:

Švedov, I. A. Projection spectra and the completeness of topological spaces. Dokl. Akad. Nauk SSSR **128** (1959), 250–252. (Russian)

The method of projection spectra [P. Alexandroff, Ann. of Math. (2) **30** (1928), 101–187] is used to study the interrelation between families Σ of open coverings of a T_1-space R. A class $\Sigma = \{\alpha, \beta, \cdots\}$ of complexes, together with mappings $\pi_\beta{}^\alpha$, is a 'spectrum' if (1) for each simplex $t \in \beta$, $\pi_\alpha{}^\beta t$ is a class (perhaps void) of vertices of α whose finite subsets are simplexes, and (2) $\pi_\alpha{}^\beta(\pi_\beta{}^\gamma e) \subset \pi_\alpha{}^\gamma e$ for each vertex e of γ ($\alpha, \beta, \gamma \in \Sigma$). For such a Σ, a limit space $L(\Sigma)$ can be readily defined. The limit space $L(\Sigma)$ has not only a natural topology, but also a natural Σ-system, i.e., a family of open coverings the stars of which at each point form a basis there. A Σ-system Σ_1 on R_1 is an 'extension' of another, Σ, on R, if R_1 is an extension of R and

$$\Sigma_1 = \{\{(R \cap \nu')^{-\prime} : V \in \alpha\} : \alpha \in \Sigma\}.$$

Some results can now be stated. Given a Σ-system Σ on R, $L(\Sigma)$ provides the greatest extension of Σ. The system Σ of all open coverings of R yields $L(\Sigma)$ whose space is homeomorphic to R. The class K of all finite coverings yields Wallman's $\omega(R)$. *R. Arens* (Los Angeles, Calif.)

7494:

Freud, G. Ein Beitrag zu dem Satze von Cantor und Bendixson. Acta Math. Acad. Sci. Hungar. **9** (1958), 333–336.

Let R be a T_1 topological space. Let $\mathfrak{A}$ be a non-empty class of subsets of R such that: (1) Every subset of an element of $\mathfrak{A}$ belongs to $\mathfrak{A}$; (2) The union of two elements of $\mathfrak{A}$ belongs to $\mathfrak{A}$. Let $C \subset R$. c is called an $\mathfrak{A}$-limit point of C if for every open set U_c containing c we have

$$(C \cap U_c) - c \notin \mathfrak{A}.$$

Let C' be the set of all the $\mathfrak{A}$-limit points of C. The $\mathfrak{A}$-closure $\bar{C}$ of C is defined to be $C \cup C'$. $\bar{C}$ satisfies the axioms of Kuratowski for a T_1 topological space. Let $\mathfrak{A}^*$ be a class of subsets of R satisfying in addition to (1) and (2) also the following properties: (3) The subsets of R consisting of only one point belong to $\mathfrak{A}^*$; (4) If to every $a \in A$ there is a neighborhood U_a of a for which $A \cap U_a \in \mathfrak{A}^*$, then $A \in \mathfrak{A}^*$. It is shown (among other results) that: Every $\mathfrak{A}^*$-closed set $C \subset R$ is the union of an $\mathfrak{A}^*$-perfect set and an $\mathfrak{A}^*$-set and conversely. Every $\mathfrak{A}^*$-perfect set is also perfect in the original topology of R. Applications of the theorems are also given.

Hing Tong (Middletown, Conn.)

7495:

Ivanov, A. A. Contiguity relations on topological spaces. Dokl. Akad. Nauk SSSR **128** (1959), 33–36. (Russian)

The theory of faithful compactifications [V. M. Ivanova and A. A. Ivanov, same Dokl. **127** (1959), 20–22; MR **21** #5943] is generalized from T_1 spaces to arbitrary spaces, and a beginning is made on the classification of all faithful extensions. *J. Isbell* (Lafayette, Ind.)

7496:

Ward, L. E., Jr. A fixed point theorem for chained spaces. Pacific J. Math. **9** (1959), 1273–1278.

Borsuk proved [Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 17–20; MR **16**, 275] that every hereditarily unicoherent arcwise connected metric continuum has the fixed-point property. The reviewer proved [Amer. J. Math. **68** (1946), 479–494; MR **8**, 49] that every arcwise connected metric space in which every monotone collection of arcs is contained in an arc has the fixed-point property. The author generalizes both of these results. To specialize his result to metric spaces for brevity, a ray of a space X with endpoint e is defined to be the one-to-one continuous image of the non-negative reals, e being the image of 0. If R is a ray with end point e, and x is in R, $A(R, x)$ denotes Cl $(R - [e, x])$, and K_R denotes $\bigcap \{A(R, x); x \in R\}$. Theorem: Suppose that the arcwise connected metric space X contains no simple closed curve and has the property that for each ray R, the set K_R is non-empty and has the fixed-point property. Then X has the fixed-point property. The author's result actually is true for Hausdorff spaces and generalized arcs, i.e., continua with just two non-end points. *G. S. Young* (New Orleans, La.)

7497:

Sandor, St.; et Wexler, D. Sur la stabilité dans les systèmes dynamiques. Rev. Math. Pures Appl. **3** (1958), 325–328.

This paper is concerned with a theorem on "composed stability" of a stationary dynamical system. The rather technical statement is best understood from one of the corollaries: Let $p_t = f(t; p)$ describe a stationary flow in a metric space R; $t \geqq 0$, $p, f \in R$. Let $A \subset B \subset C \subset R$ be three compact sets, invariant for $t \to +\infty$. If then A is asymptotically stable with respect to B and B stable with respect to C, then A is stable with respect to C. This is a consequence of a more general theorem proven in the paper. *J. Moser* (Cambridge, Mass.)

7498:

Bass, Robert W. Zubov's stability criterion. Bol. Soc. Mat. Mexicana (2) **4** (1959), 26–29.

Let M be a closed invariant set of a dynamical system. Zubov [*Metody A. M. Lyapunova i ih primenenie*, Izdat. Leningrad Univ., Moscow, 1957; MR **19**, 275] has stated erroneously that a necessary and sufficient condition for the stability of M is that no path from outside M possess an α-limit point in M. In this paper the author gives a necessary and sufficient condition for the stability of M, using the concept of saddle set.

M. M. Peixoto (Rio de Janeiro)

ALGEBRAIC TOPOLOGY

See also 7268.

7499:

Kelly, G. M. Single-space axioms for homology theory. Proc. Cambridge Philos. Soc. **55** (1959), 10–22.

From the author's introduction: "Eilenberg and Steenrod give a set of axioms for the homology theory of pairs of spaces and their maps, and prove that these axioms are categorical on triangular pairs. Here we give a set of axioms for the homology theory of single spaces and their maps, that is, for absolute rather that relative homology. This axiomatization is shown to be essentially equivalent to that of Eilenberg and Steenrod, the relative homology groups being suitably defined in terms of the absolute groups." The join XY of two spaces X and Y is defined in a new way so that the join operation is associative and $A \subset X$ implies $AY \subset XY$. A detailed development of the properties of this join is given.

Let $f: X \to Y$ be a map inducing $f'': X(p \cup q), p, q \to Y(r \cup s), r, s$, a map of the suspension of X into that of Y. The definition of an absolute homology theory assigns (i) to each integer α an abelian group $H_\alpha(X)$, (ii) an induced homomomorphism $f_\sharp: H_\alpha(X) \to H_\alpha(Y)$, (iii) a homomorphism Δ:

$$H_\alpha(X(p \cup q)) \to H_{\alpha-1}(X);$$

these satisfying: (AH1) if $g: Y \to Z$ then $(gf)_\sharp = g_\sharp f_\sharp$, (AH2) $f_\sharp \Delta = \Delta f_\sharp''$, (AH3) Δ is an isomorphism, (AH4) if $A \subset X$ then the inclusion maps $i: A \to X$ and $j: X \to X \cup Ap$ induce homomorphisms which give an exact sequence (note only two parts of the usual exactness axiom are retained), (AH5) $H_\alpha(\emptyset) = 0$ if $\alpha \neq -1$.

Given a relative homology theory $\mathscr{R}$ with homology groups $H_\alpha(X, A)$, from this an absolute homology theory $\mathscr{R}'$ with homology groups $\hat{H}_\alpha(X)$ is defined and from this a relative homology theory $\mathscr{R}''$ with homology groups $\hat{H}_\alpha(X, A)$ is defined. These definitions are:

$$\hat{H}_\alpha(X) = H_{\alpha+1}(Xp, X),$$

with the Δ operator defined by the sequence

$$H_\alpha(X(p \cup q)) \to H_\alpha(X(p \cup q), Xp) \xleftarrow{\approx} H_\alpha(Xq, X) \xrightarrow{\partial_\sharp} H_{\alpha-1}(X);$$

$\hat{H}_\alpha(X, A) = \hat{H}_\alpha(X \cup Ap)$, with the boundary operator in $\mathscr{R}''$ given by the sequence

$$\hat{H}_\alpha(X \cup Ap) \to \hat{H}_\alpha(Xq \cup Ap) \xleftarrow{\approx} \hat{H}_\alpha(A(p \cup q)) \to \hat{H}_{\alpha-1}(A) \to \hat{H}_{\alpha-1}(A \cup p).$$

The author shows that $\mathscr{R}$ and $\mathscr{R}''$ are equivalent.

A. J. Goldstein (Murray Hill, N.J.)

7500:

Venkov, B. B. Cohomology algebras for some classifying spaces. Dokl. Akad. Nauk SSSR **127** (1959), 943–944. (Russian)

E. Golod [same Dokl. **125** (1959), 703–706; MR **21** #3473] has shown, using the Hochschild-Serre spectral sequence, that the cohomology algebra of a finite p-group over Z_p is finitely generated (fg). This result is generalized as follows. Let G be a closed subgroup of the general linear group GL (n, C), let B_G be its classifying space and suppose that $H^i(B_G; Z_p)$ is fg for $0 \leq i \leq 4n^2$. Then (theorem 1) $H^*(B_G; Z_p)$ is fg. This gives Golod's theorem since $H^*(B_G; Z_p)$ is the cohomology algebra of the abstract group G if G is discrete; it also has the consequence that $H^*(B_G; Z_p)$ is fg if G is a compact Lie group. Theorem 2 gives more precise information about $H^*(B_G; Z_p)$ under the hypotheses of theorem 1; namely, its Poincaré series is that of a rational function whose denominator is a product $\prod_k (1-t^k)^{\alpha_k}$, where $\sum_k \alpha_k \leq n$.

If G is a closed subgroup of the connected group K there is a canonical fibre map $B_G \to B_K$ with fibre K/G; this leads to a spectral sequence I. The fibration $K \to K/G$ is induced by a map $K/G \to B_G$ which is equivalent to a fibre map with fibre K; this leads to a spectral sequence II. Theorem 1 is proved by using the sequences I and II together with the known structure of $H^*(B_{U(n)}; Z_p)$ where $U(n)$ is the unitary group.

P. J. Hilton (Birmingham)

7501:

Nakamura, Tokusi. Equivalence between two definitions of the cohomology operations. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **9** (1959), 1–16.

There are two different approaches to the subject of cohomology operations. On the one hand, one can define a cohomology operation relative to dimensions q, r and coefficient groups A, B and then prove easily that such cohomology operations are in 1-1 correspondence with the elements of the cohomology group $H^r(A, q; B)$ of an Eilenberg-MacLane complex. On the other hand, Steenrod has given a constructive method of deriving cohomology operations from the cohomology of the symmetric group [see Comment. Math. Helv. **31** (1957), 195–218; MR **19**, 1069]. Steenrod has emphasized the importance of the problem of establishing a direct relation between these two methods which would permit one to translate a proof of a formula in one theory into a proof of the corresponding formula in the other theory. The present paper is addressed to this problem.

The author's first result is an algebraic analogue of a well known theorem of Dold and Thom [Ann. of Math. (2) **67** (1958), 239–281; MR **20** #3542]. He shows that the Eilenberg-MacLane complex $K(Z, q)$ contains a subcomplex $K(N, q)$ which is of the same homotopy type as $K(Z, q)$ and such that $K(N, q)$ is the infinite symmetric product of a sub-complex $K(N, q, 1)$ which is homotopically equivalent to the singular complex of a q-dimensional sphere. Moreover, $K(N, q)$ is the union of an ascending sequence of sub-complexes $K(N, q, n)$, $n = 1, 2, 3, \cdots$, such that $K(N, q, n)$ is the n-fold symmetric product of $K(N, q, 1)$.

In order to state the author's second result, let π denote the symmetric group of degree n, and let W denote the standard (non-homogeneous) π-free acyclic complex. Let $M = M(q, 0)$ denote the standard elementary cochain

complex such that $C^q(M)$ is infinite cyclic, and $C^i(M)=0$ for $i\neq q$. Let M^n denote the tensor product of n copies of M on which π operates by permuting the factors. Then the author proves that the chain complex

$$K(N, q, n)/K(N, q, n-1)$$

is homotopically equivalent to a direct summand of $W\otimes_\pi M^n$. The importance of this result stems from the fact that the cohomology classes on $W\otimes_\pi M^n$ correspond to the cohomology operations whose domain is the q-dimensional integral homology group by Steenrod's method.

The author promises applications in subsequent papers.

W. S. Massey (Providence, R.I.)

7502:

Toda, Hirosi. A topological proof of theorems of Bott and Borel-Hirzebuch for homotopy groups of unitary groups. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 103–119.

Let $SU(m)$ denote the special unitary group, $SU=\bigcup_m SU(m)$, $M=K(Z, 2)$, and EM denote the suspension of M. The CW-complex SU has the following well-known properties: (U_1) it is simply connected; (U_2) it is a homotopy associative H-space; (U_3) its integral cohomology ring is an exterior algebra generated by elements e_i of dimension $2i+1$, $i=1, 2, \cdots$; and (U_4) there is a map $f: EM\to SU$ such that the induced map f^* in integral cohomology is onto.

The author proves (the main theorem) that if a space X has the properties $(U_1), \cdots, (U_4)$, then the second loop space of X with the fundamental group killed, Ω^2X/π_1, also has these properties. This gives an alternate proof of the Bott periodicity theorem for the homotopy groups of SU; i.e., $\pi_{2n+1}(SU)\simeq Z$ and $\pi_{2n}(SU)\simeq 0$ [Bott, Michigan, Math. J. **5** (1958), 35–61; MR **21** #1589].

His argument proceeds by applying the homology analogue of the weak form of the Borel transgression theorem [Borel, Ann. of Math. (2) **57** (1953), 115–207; MR **14**, 490], proved here by use of the comparison theorem for spectral sequences. This is sufficient to yield the Hopf algebra structure of the integral homology of ΩX. The author then computes the Hopf algebra structure of the integral cohomology of ΩX, and another spectral sequence argument, again using the comparison theorem, then yields properties (U_1), (U_2) and (U_3).

Basic to obtaining property (U_4) is the fact that for any H-space Y there is a canonical map $Y\circ Y\to EY$, where $\circ$ denotes join. For the H-space M there is then the composition

$$\zeta: E^2M = E(S^2 * M)\to E(M * M)\to M\circ M\to EM.$$

The author then shows that the map

$$EM\to\Omega^2E^2M\to\Omega^2EM\to\Omega^2X/\pi_1$$

satisfies condition (U_4).

It is also shown that the composition

$$S^{2n+1} = E^{2n-1}S^2\to E^{2n-1}M\to E^{2n-2}M\to\cdots\to E^2M\to EM\to SU$$

generates $\pi_{2n+1}(SU)$ and the homology image of the generator $\iota_{2n+1}\in H_*(S^{2n+1}; Z)$ is $\pm n!$ times the generator $\tilde{c}_n\in H_*(SU; Z)$. Having determined the Hurewicz homomorphism, he then easily computes the unstable group $\pi_{2n}(SU(n))\simeq Z_{n!}$, $n\geq 2$. Finally, the author applies these methods to compute the groups $\pi_{2n+1}(SU(n))$ and $\pi_{2n+2}(SU(n))$.

The main theorem is somewhat weaker than the important conclusion of Bott that B_u and ΩSU are of the same homotopy type. A slight variant of the argument made in the paper does however yield that result. It may also be remarked that the conditions $(U_1), \cdots, (U_4)$ can be shown to characterize SU up to homotopy type.

E. Dyer (Chicago, Ill.)

7503:

Eckmann, B.; and Hilton, P. J. On the homology and homotopy decomposition of continuous maps. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 372–375.

In previous papers [cf. C. R. Acad. Sci. Paris **246** (1958), 2444–2447, 2555–2558, 2991–2993; MR **20** #6694, #6695, #6696] the authors have introduced the homotopy and homology groups of a map $f: X\to Y$. {In a more familiar terminology, these are the groups of the pair (C_f, X) where C_f is the mapping cylinder of f; the authors, however, have no need of this space.}

In the present paper, given a map $f: X\to Y$ the authors describe a factorisation

$$X\xrightarrow{f_1} X_1\xrightarrow{f_2} X_2\to\cdots\xrightarrow{f_k} X_k\xrightarrow{f_k} Y$$

of such a map, where f_k is the result of "killing" the lowest k homology-groups of f, and the j_i are co-fibrations, i.e., inclusions satisfying a homotopy extension property. The successive spaces $X_1, X_2, \cdots$ are the result of modifying successively the homology groups of X so as to get those of Y. If X is a point, one obtains the "homology decomposition of Y", previously described by the authors.

The dual notion is the "homotopy decomposition" of f, a factorisation

$$X\xrightarrow{f^k} Y^k\xrightarrow{q^k}\cdots\xrightarrow{q^2} Y^1\xrightarrow{q^1} Y$$

of f, q^r being a fibre-map with fibre $K(\pi_{m_r}(f), m_r-1)$, where $\pi_{m_r}(f)$ is the rth non-vanishing homotopy group of f. This "generalised Postnikov decomposition" of f was previously described by J. C. Moore, using an entirely different, semisimplicial approach. The authors' method allows a good description of the invariants of the decomposition, which becomes the classical one of X when Y is a point.

The authors give one application of the homology decomposition: Theorem: Let X be a 1-connected polyhedron with exactly m non-vanishing homology-groups. Then the Lusternik-Schnirelman category of X is $\leq m+1$.

V. Gugenheim (Baltimore, Md.)

7504:

Malm, D. G. Concerning the cohomology ring of a sphere bundle. Pacific J. Math. **9** (1959), 1191–1214.

"This paper is concerned with the problem of determining the cohomology ring of an orientable fibre space whose fibre is a sphere, in terms of the cohomology ring of the base space and invariants of the fibre space.

"When the fibring sphere is of even dimension $k-1$, an invariant P in the $(2k-2)$-dimensional cohomology group of the base space is defined, which is closely related to one of the Pontrjagin characteristic classes if the fibre space is

a fibre bundle. If the $(2k-2)$-dimensional cohomology group of the base space B has no elements of order two, then two $(k-1)$-sphere spaces over B with the same Stiefel-Whitney classes W_k and W_{k-1} and the same invariant P have isomorphic integral cohomology rings.

"In the other case, when k is even, if $H^{2k-2}(B, Z)$ has no two-torsion, then two $(k-1)$-sphere spaces over B with the same Stiefel-Whitney classes W_k, W_{k-1}, and W_{k-2} have isomorphic integral cohomology rings.

"If $H^{2k-2}(B, Z)$ has elements of order two the situation seems to be more complicated and no results are obtained. Also, the problem of determining the cohomology ring with mod 2 coefficients is not touched upon here.

"The method is based upon the algebraic mapping cylinder of the map $x \to xU$, where U is Thom's class, and thus parallels Thom's construction of the Gysin sequence using the mapping cylinder." (From the author's introduction)

The author points out that it is misleading to say that "the cohomology ring of the total space depends on the cohomology of the base space and certain characteristic classes." Just exactly what aspects of the homotopy type of the base space it does depend on seems like a very complicated question. Certainly various higher order cohomology operations are needed.

W. S. Massey (Providence, R.I.)

7505:

Borel, Armand. Fixed points of elementary commutative groups. Bull. Amer. Math. Soc. **65** (1959), 322–326.

An elementary $[p]$-group is the direct product of a finite number of groups Z_p when p is a prime and a toral group when $p=0$. Let (G, X) be a transformation group in which G is an elementary $[p]$-group, and let F be the set of points x in X such that $gx=x$ for $g \in G$. Assume that X is locally connected. (1) If $H^*(X; K_p)$ is finite (where K_p is a field of characteristic p) and if X is totally non-homologous to zero mod p in X_G, where X_G is the bundle $X \times E_G \to B_G$ with typical fiber X, $E_G \to B_G$ being the universal bundle of G, then

$$\dim H^*(X; K_p) = \dim H^*(F; K_p).$$

In particular, $F \neq \emptyset$. (2) Let $G_1, G_2, \cdots$ be the subgroups of G of index p (of codimension 1 if $p=0$). If X is a cohomology n-sphere mod p so that F is a cohomology r-sphere mod p and the fixed-point set of G_i is a cohomology n_i-sphere mod p, then the integers n, r, n_i satisfy the relation $n-r=\sum (n-n_i)$. A compact connected $2n$-manifold X is called homologically Kählerian if there exists a class $Q \in H^2(X; K_0)$ such that multiplication by Q^{n-s} is an isomorphism of $H^s(X; K_0)$ onto $H^{2n-s}(X; K_0)$, $i \leqq s \leqq n$. (3) If G is toral and X is homologically Kählerian then (a) if $F \neq \emptyset$, $\dim H^*(F; K_0) = \dim H^*(X; K_0)$; (b) if $H^1(X; K_0)=0$, then $F \neq \emptyset$; (c) if X has no torsion, F has no p-torsion. (4) A theorem giving a sufficient condition that every elementary $[p]$-subgroup of a compact connected Lie group K be conjugate to a subgroup of a given closed subgroup of K. (5) Assume that $H^*(X; K_p) = K_p[x]/x^{s+1}$ where the degree of x is even if $p \neq 2$. Then $\dim H_c^*(F; K_p)$ (compactly supported cohomology) equals $s+1$ under various specified conditions such as $(p, s+1)=1$. (6) A theorem similar to (2) for local cohomology in a generalized manifold. *P. A. Smith* (New York, N.Y.)

7506:

Stojaković, Mirko. Sur la fonction chromatique. Univ. Beograd. Godišnjak Filozof. Fak. Novi Sad **2** (1957), 337–356. (Serbo-Croatian. French summary)

A graph is 2-chromatic if and only if all closed circuits are even. Elaborations of this fact. Proof that a certain naive method for coloring is useless. Coloring maps with 17 regions by alternating chains.

P. Ungar (New York, N.Y.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 7287, 7288.

7507:

★Kreyszig, Erwin. Differential geometry. Mathematical Expositions, No. 11. University of Toronto Press, Toronto, 1959. xiv+352 pp. $8.50.

This is a standard textbook on the differential geometry of curves and surfaces lying in three-dimensional Euclidean space. Surfaces are treated by the methods of the tensor calculus, and so the book gives an introduction to tensor calculus and Riemannian geometry. The chapter headings are as follows: (I) Preliminaries. (II) Theory of curves. (III) Concept of a surface. First fundamental form. Foundations of tensor calculus. (IV) Second fundamental form. Gaussian and mean curvature of a surface. (V) Geodesic curvature and geodesics. (VI) Mappings. (VII) Absolute differentiation and parallel displacement. (VIII) Special surfaces. There are numerous figures which are unusually well done.

The book is written in the spirit of differential geometry in the small and contains only passing references to manifolds and similar topics in contemporary differential geometry. It is to be regretted that the exposition makes no use of recent algebraic concepts such as the algebraic definition of a tensor, tensor products, and dual vector spaces. Students meet these notions in their algebra courses and should see their applications in other fields. In the definition of g^{ij} the author even fails to mention that he is doing nothing more elaborate than constructing the inverse of a matrix.

It seems to be almost impossible to write a book on differential geometry in which all theorems are correctly stated, but this book has a much higher standard than most. Frequently the hypotheses are stronger than are absolutely necessary, but it is clear that these were chosen to avoid distracting the reader with troublesome special cases. As an example, the theory of curves assumes that the curvature is strictly positive for all values of the parameter, and hence straight lines and curves with inflections are excluded from the theory. It should be noted that the orientation of a curve is not properly defined although it is used systematically in the development. The author is to be complimented on his treatment of the theory of contact, which is far clearer than that in the classical treatises. *C. B. Allendoerfer* (Seattle, Wash.)

7508:

de Rham, Georges. Sur les courbes limites de polygones obtenus par trisection. Enseignement Math. (2) **5** (1959), 29–43.

Let $S_0, S_1, \cdots, S_n$ be the $n+1$ vertices of a broken-polygon with n edges (i.e., an $n+1$-gon with one edge deleted). Let new points S'_{2i} and S'_{2i+1} divide the edge S_iS_{i+1} into three segments $S_iS'_{2i}$, $S'_{2i}S'_{2i+1}$, $S'_{2i+1}S_{i+1}$ proportional to the given positive numbers β_1, α, β_2 whose sum is 1. The points $S'_0, S'_1, \cdots, S'_{2n-1}$ are the vertices of a new broken-polygon P' of $2n-1$ edges. Starting with a broken-polygon P_0 with 2 edges, repetition of this "trisection" process leads to a broken-polygon P_n of 2^n+1 sides (P_{n+1} obtained from P_n by "trisection", where the numbers β_1, α, β_2 remain fixed). The author obtains properties of the curve C which is the limit curve of the sequence P_n. *W. Moser* (Winnipeg, Man.)

7509:

Hsiao, E. K. Another proof for two properties of the generators of a ruled surface. Amer. Math. Monthly **66** (1959), 707–709.

Si une surface réglée est rapportée à ses génératrices $v=\text{const}$ et à leurs trajectoires orthogonales $u=\text{const}$, on doit à C. E. Weatherburn [C. E. Weatherburn, *Differential geometry of three dimensions*, vol. II, Cambridge Univ. Press, London, 1930; p. 45] la formule suivante: $\partial(\sqrt{-K})/\partial u = -2(\sqrt{-K})\operatorname{div}\mathbf{a}$, où $\operatorname{div}\mathbf{a}$ et K représentent respectivement la courbure géodésique des trajectoires orthogonales et la courbure totale. On doit également à Ram Behari [Tôhoku Math. J. **46** (1939), 41–43; MR **1**, 170] le théorème suivant: si le long d'une génératrice d'une telle surface la courbure moyenne varie en restant inversement proportionnelle à $\sqrt{G}$, G étant le troisième coefficient de la première forme fondamentale, la courbure totale reste constante le long de chaque trajectoire orthogonale. Le présent auteur donne de ces deux propriétés des démonstrations différentes.

F. Şemin (Istanbul)

7510:

Ramacci, Maria Gabriella. Su alcune notevoli rappresentazioni analitiche delle curve gobbe. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 469–480.

Sia Γ una generica curva di S_3 descritta dal punto $P(t)$ $(t_1<t<t_2)$. Considerati i vettori derivati il vettore $\bar{P}$ è combinazione lineare dei vettori indipendenti $\dot{P}$, $\ddot{P}$, $\dot{P}\wedge\ddot{P}$. La Γ è quindi soluzione di un'equazione vettoriale del 3° ordine; e viceversa una tale equazione determina una classe di curve. Oggetto del lavoro è la riduzione della detta equazione, per cambiamenti del parametro t, ad una forma nella quale i coefficienti (ed il nuovo parametro) assumano particolari significati geometrici legati ad invarianti delle curve per il gruppo delle congruenze oppure delle similitudini. *C. Longo* (Parma)

7511:

Drăgilă, P. Correspondance entre deux surfaces par des faisceaux de tangentes parallèles. Proc. Amer. Math. Soc. **10** (1959), 366–368.

Rectification of a preceding statement on the same subject [cf. same Proc. **9** (1958), 855–859; MR **21** #2252].

E. Bompiani (Rome)

7512:

Löbell, Frank. Zusammenhänge zwischen Vektoranalysis und Krümmungstheorie der Kurvenkongruenzen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. 1958, 73–79.

Given a vector-field (in a 3-dimensional Euclidean space) and the congruence of curves determined by it, F. Emde gave formulas expressing the divergence and the curl of the field. A new proof of these formulas is offered, together with new applications. *E. Bompiani* (Rome)

7513:

Gheorghiu, O. Em.; et Crstici, B. Sur quelques objets géométriques à deux composants. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 311–331. (Romanian. Russian and French summaries)

This paper deals with the linear, non-differential geometrical objects in X_n $(n\geq 2)$ having two components Ω_1, Ω_2 which transform like $(x, y)\to(\bar{x}, \bar{y})$ in the formulae

$$\bar{x} = a_1x+b_1,$$

$$\bar{y} = a_2y+a_2^1x+b_2.$$

This condition leads to a set of functional equations between the coefficients in the law of transformation, which are assumed to be continuous (but not necessarily differentiable) functions of their arguments. These arguments are, of course, the old coordinates x^i and the new coordinates $\bar{x}^s$ together with their derivatives with respect to the x^i up to the rth order, where r is the "class" of the corresponding component.

The consideration of the above-mentioned system of functional equations leads to the following results. (1) The explicit laws of transformation in the case Ω_1 and Ω_2 of class zero, and in the case Ω_1 of class zero and Ω_2 of class one. (2) No geometrical objects of the type discussed exist for which Ω_1 is of class zero and Ω_2 of class $v\geq 2$. (3) There exist five different types for which Ω_1 and Ω_2 are of class one. The explicit laws of transformation of these are given.

In all these laws of transformation there appear 5 arbitrary functions of n arguments, and 3, 4 or 5 arbitrary constants. *R. Blum* (Saskatoon, Sask.)

7514:

Gheorghiu, Oct. Em. Sur les objets géométriques associés à un système linéaire d'équations aux dérivées partielles du premier ordre. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 145–158. (Romanian. Russian and French summaries)

The system of equations is

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}+\sum_{i=1}^{3} a_{i1}u^i = 0,$$

$$\frac{\partial u_3}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_1}{\partial x_3}+\sum_{i=1}^{3} a_{i2}u^i = 0,$$

$$\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}+\frac{\partial u_3}{\partial x_3}+\sum_{i=1}^{3} a_{i3}u^i = 0.$$

The author finds the group of transformations of the independent variables leaving the system invariant if the u_i are either invariant or relative invariant functions, and the group of transformations of the dependent variables of the form $u_i=\sum_{j=1}^{3} f_{ij}(x)\bar{u}_j$ leaving the system invariant;

he finds various types of geometric objects for these groups. He also points out that the hypercomplex variable $\omega = x_1 + \theta x_2 + \theta^2 x_3$, where $\theta^3 = 1$, is naturally related to the system. *J. B. Crabtree* (Hoboken, N.J.)

7515:

Hosszú, Miklós. Functional equations and algebraic methods in theory of geometric objects. I, II, III. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 149–162, 237–253, 333–346. (Hungarian)

This is a somewhat more detailed Hungarian version of the author's paper in Publ. Math. Debrecen **5** (1958), 294–329 [MR **21** #3518], and also contains some new results. *J. Aczél* (Debrecen)

7516:

★Широков, П. А.; и Широков, А. П. Аффинная дифференциальная геометрия. [Širokov, P. A.; and Širokov, A. P. Affine differential geometry.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 319 pp. 9.85 rubles.

Das vorliegende Werk behandelt alle Teile der modernen affinen Differentialgeometrie. Es enthält auch einige von den neuesten Resultaten auf diesem Gebiet, die sich sonst nur in den Zeitschriften befinden. Die Grundlage des Werkes bilden die hinterlassenen Manuskripte des erstgenannten Verf., P. A. Širokov, die sich auf seine Vorlesungen aus dem Jahre 1938 beziehen. Der zweitgenannte Verf. hat dieses Material geordnet und besonders mit Rücksicht auf die neuen Resultate bedeutend ergänzt. Er hat auch eine reichhaltige Bibliographie, enthaltend 401 Arbeiten, Einzeldarstellungen und Werke bis einschliesslich zum Jahre 1957, zusammengestellt.

Das Buch enthält nebst Einführung zwei Teile (Kurventheorie, S. 75–113; Flächentheorie, S. 114–274) und einen Anhang von A. P. Norden "Über die innere Geometrie zweiter Art auf einer Hyperfläche des affinen Raumes". Unter anderen sonst in einem Lehrbuch üblichen Fragen kommen noch zur Behandlung: relative Flächentheorie, eine Methode der Untersuchung der Kongruenzen von H. G. Gasparian [Nauč. Trudy Erevan. Univ. **30** (1950), 21–64] und die Integralsätze der affinen Flächentheorie von Grotemeyer [Arch. Math. **3** (1952), 38–43; MR **14**, 788]. Ausserdem benützen die Verf. neben den Begriffen der metrischen inneren Geometrie auch die Begriffe der inneren Geometrien der ersten und der zweiten Art, die von A. P. Norden eingeführt wurden.

Die Verf. verwenden folgerichtig die Tensormethoden in ihrer Darstellung. Das Buch ist mit Sachverzeichnis versehen und technisch gut ausgestattet. Es ist sehr geeignet zur Einarbeitung in dieses Gebiet und enthält gelöste Beispiele und Aufgaben.

T. P. Anđelić (Belgrade)

7517:

Longo, Carmelo. Le calotte del secondo ordine di S_3, con centro assegnato. Riv. Mat. Univ. Parma **8** (1957), 49–58.

The surface caps of the second order (in the projective space S_3) which have a common center are studied. Bompiani's representation of the regular caps by means of linear spaces [Ann. Mat. Pura Appl. (4) **22** (1943), 1–32; MR **8**, 349] is attained here more directly by considering the linear systems of quadrics determined by the caps. From the grassmanian coordinates for a cap the author characterises and studies a surface V_8 whose points represent the caps, determining the order of the V_5. This V_5 is a minimum model for the caps in the sense of Severi. *A. Schwartz* (New York, N.Y.)

7518:

Marcus, F. Quelques observations sur les surfaces de Godeaux et Demoulin, et sur une classe de surfaces signalée par Bompiani. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 462–469.

The author writes out analytic conditions that characterize surfaces of Godeaux and Demoulin and also expressions for the projective arc for an asymptotic curve, thus pointing out that for a surface of Godeaux [Demoulin] the curves of one family [both families] of asymptotics correspond with equality of arc. The other minimal-projective surfaces which have this property are determined. Some surfaces of Bompiani [Fubini and Čech, *Geometria proiettiva differenziale*, Zanichelli, Bologna, 1926, second appendix] have the property that the asymptotics of one family correspond with proportionality of projective arc. The surfaces discussed here are special cases. *A. Schwartz* (New York, N.Y.)

7519:

Prvanovitch, Mileva. Relative Frenet formulas for curves in a subspace of a Riemannian space. Tensor (N.S.) **9** (1959), 190–204.

The first relative curvature vector and the first relative curvature of a curve C in a subspace V_n of a Riemannian space V_m were first introduced by the reviewer [Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 3–9; MR **20** #309]. By means of these concepts the author defines the relative intrinsic derivative of the unit tangent vector of C in V_n in V_m and derives two sets of equations generalizing Frenet formulas for C in V_n in V_m in terms of relative pth curvatures and relative pth curvature vectors ($p = 1, 2, \cdots, n-1$) of the first kind and those of the second kind. The following theorem is proved. A curve C in V_n in V_m is uniquely determined if the relative pth curvatures ($p = 1, 2, \cdots, n-1$) of the first [second] kind are given as functions of s and of the unit vectors generating the $m-n$ congruences and if the direction of C together with the relative pth curvature vectors of the first [second] kind is given at one point.

T. K. Pan (Norman, Okla.)

7520:

Toponogov, V. A. Riemannian spaces containing straight lines. Dokl. Akad. Nauk SSSR **127** (1959), 977–979. (Russian)

Let R^m be an m-dimensional complete Riemannian manifold with non-negative curvature which is assumed for the sake of brevity to be differentiable of class ∞. Then the following theorem is proved: If in R^m there exist geodesic lines of which every segment is the shortest one, then R^m must represent itself as the direct metrical product of a straight line and an $(m-1)$-dimensional Riemannian manifold R^{m-1} with non-negative curvature, i.e., $R^m = E^1 \times R^{m-1}$. In order to prove this theorem the author prepares 6 lemmas which are proved pure-geometrically, on making use of the notion of limit, on the

basis of the following two theorems of the author [same Dokl. **115** (1957), 674–676; MR **19**, 979]: (1) In the manifold R^m the convexity condition of A. D. Aleksandrov [Mat. Sb. (N.S.) **1** (**43**) (1936), 307–322] is satisfied; (2) angles of an arbitrary triangle in R^m composed of shortest arcs are not less than the corresponding angles of the flat euclidean triangle with the sides of the same length. The theorem is generalized to the case $R^m = E^k \times R^{m-k}$.

A. Kawaguchi (Sapporo)

7521:

Hartman, Philip. On isometries and on a theorem of Liouville. Math. Z. **69** (1958), 202–210.

If two non-singular Riemann metrics of class C^σ, $\sigma = 1, 2, \cdots$, are isometric by virtue of a mapping of class C^1, then the mapping is necessarily of class $C^{1+\sigma}$. If (*) $v = v(u)$ is a C^1 mapping with non-vanishing Jacobian which maps the Euclidean metric

$$ds^2 = \gamma^2[(dv^1)^2 + \cdots + (dv^q)^2],$$

$q \geq 3$, into a conformal metric

$$ds^2 = \gamma^2[(du^1)^2 + \cdots + (du^q)^2],$$

where $\gamma = \gamma(u) > 0$, then (*) is a Möbius transformation and, hence, is analytic. (The latter statement is an extension of a theorem of Liouville.) Both these results are obtained as applications of the following assertion concerning the smoothness of solutions of overdetermined systems of partial differential equations. Let

$$u = (u^1, \cdots, u^q), \quad v = (v^1, \cdots, v^q);$$

let

$$f^I(u, v, v_1{}^1, v_2{}^1, \cdots, v_q{}^q) \in C^\sigma, \quad I = 1, 2, \cdots, q(q+1)/2;$$

let (*) $v = v(u)$ be of class C^1 and satisfy the system of partial differential equations

$$(**) \qquad f^I(u, v, \partial v^1/\partial u^1, \cdots, \partial v^q/\partial u^q) = 0, \qquad I = 1, 2, \cdots, q(q+1)/2;$$

in addition, let $\Delta(u) \neq 0$, where $\Delta(u)$ is the determinant of the matrix of coefficients of $\partial^2 v^i/\partial u^j \partial u^k$ in the system of $q^2(q+1)/2$ linear equations obtained by differentiating (**) formally with respect to u^k, $k = 1, \cdots, q$, and assuming $\partial^2 v^i/\partial u^j \partial u^k = \partial^2 v^i/\partial u^k \partial u^j$. Then (*) is of class $C^{1+\sigma}$. The proof of this assertion uses finite differences and the notion of strong L^2-derivatives.

A. Douglis (College Park, Md.)

7522:

Nagano, Tadashi. Homogeneous sphere bundles and the isotropic Riemann manifolds. Nagoya Math. J. **15** (1959), 29–55.

Theorem: Let M be a connected manifold with Riemannian metric of class C^2. If M is isotropic at p, then there exists a C^1-diffeomorphism δ of M onto one of the following spaces: a euclidean space, a sphere, a real, complex or quaternion elliptic space, and the octave plane; δ induces an isomorphism of the isotropy groups. —Partially this is a step in the proofs of J. Tits [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1955), no. 3; MR **17**, 874] and H. Freudenthal [Math. Z. **63** (1956), 374–405; MR **18**, 591] of a larger theorem characterizing the classical geometries, which is acknowledged by the author. His proof does not explicitly use Lie group classification. The main tools are fibre bundles, used in order to determine all homogeneous sphere-bundles homeomorphic to spheres. As to the extension of the map from the isotropy spheres to the manifold, it is more akin to Tits' method than to Freudenthal's because of the use of geodesics. Lie group classification is implicitly used through Montgomery and Samelson's results [Ann. of Math. (2) **44** (1943), 454–470; MR **5**, 60] and those of A. Borel [C. R. Acad. Sci. Paris **230** (1950), 1378–1380; MR **11**, 640].

H. Freudenthal (Utrecht)

7523:

Helgason, Sigurður. On Riemannian curvature of homogeneous spaces. Proc. Amer. Math. Soc. **9** (1958), 831–838.

The author gives new proofs for the fact that the sectional curvature of a symmetric Riemannian space is ≥ 0 in the compact case and ≤ 0 in the irreducible non-compact case, and for the fact that the sectional curvature of a homogeneous space of a compact Lie group in the induced metric is ≥ 0. The basis of the proof is an elegant formula describing the differential of the exponential map of a symmetric space G/K in terms of the Lie algebras of G and K, generalizing the expression of the Maurer-Cartan forms in canonical coordinates (the latter is also proved in a very clear form). The curvature $K(S)$ on a two-dimensional subspace S of the tangent space of any point is identified with $(-3/2)\Delta f(0)$, where Δ is the Laplacian in S, and f the Jacobian of the exponential map (restricted to S). This leads to the formula $K(S) = -Q([Y, [Y, Z]], Z)$, where G defines the metric of G/K and Y, Z are two tangent vectors of G/K, interpreted as vectors in G, at the identity, orthogonal to K [cf. also K. Nomizu, Amer. J. Math. **76** (1954), 33–65; MR **15**, 468]. The main result follows by comparing Q with the Killing form. Similar computations apply to the non-symmetric case.

H. Samelson (Ann Arbor, Mich.)

7524:

Eum, Sang-Seup. On conharmonic transformations in Hermitian and Kaehlerian spaces. Kyungpook Math. J. **2** (1959), 7–15.

A special conformal transformation (which the author calls a conharmonic transformation) in the Hermitian space H_n and the Kählerian space K_n is studied. (The conharmonic transformation in the Riemannian space has been studied by Y. Ishii [Tensor (N.S.) **7** (1957), 73–80; MR **21** #1623].) The results are resuméed in theorems 1–7. Theorem 1: A necessary and sufficient condition for the conformal transformation (*) $G_{\alpha\bar\beta} = \phi^2(z, \bar z) g_{\alpha\bar\beta}$ ($g_{\alpha\bar\beta} = \partial_\beta \partial_{\bar\alpha} \phi$ in usual notation) to be conharmonic is that a vector field A_α of K_n be transformed into a vector field $B_\alpha = A_\alpha - (\log \phi)_\alpha$ of H_n and that it satisfy the condition

$$G^{\alpha\bar\beta}[B_{\alpha|\bar\beta} + (n-1)B_\alpha B_{\bar\beta}] = \phi^{-2} g^{\alpha\bar\beta}[A_{\alpha,\bar\beta} + (n-1)A_\alpha A_{\bar\beta}]$$

(",," and "|" correspond to $g_{\alpha\bar\beta}$ and $G_{\alpha\bar\beta}$ respectively). Theorem 2: A necessary and sufficient condition for a conformal transformation (*) to be a conharmonic transformation is that it satisfy the condition $H = \phi^{-2} R$ ($R = 2g^{\alpha\bar\beta} R_{\alpha\bar\beta}$, $H = 2G^{\alpha\bar\beta} H_{\alpha\bar\beta}$ in usual notation). Theorem 3: There exists no conharmonic transformation (*) under which the Ricci tensor is invariant. Theorem 4 (the main result): A necessary and sufficient condition for the conformal transformation (*) to be a conharmonic transformation is that the tensor

$$C^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} - [\delta^{\alpha}_{\delta} R_{\beta\gamma} + g_{\beta\gamma} g^{\alpha\overline{\delta}} R_{\overline{\delta}\delta}/2(n-1)]$$

remain invariant under the transformation (*). Theorem 5: If the transformation (*) is conharmonic and $C^{\alpha}_{\beta\gamma\delta}$ a zero tensor in K_n, then $R=0$ and also $H=0$ in H_n. Theorem 6: If H_n, whose sectional curvature L is the same for all possible 2-dimensional sections, is transformed by the conharmonic transformation (*) from K_n, which is a space of constant holomorphic curvature, then $L=H/n(n+1)$. Theorem 7: There exists no Hermitian space H_n which is transformed by the conharmonic transformation (*) from a Kählerian space K_n and whose holomorphic sectional curvature is all the same and not zero at all points. *T. Takasu* (Tokyo)

7525:

Haimovici, Adolf. Sur quelques invariants attachés à une direction d'un espace bi- et tridimensionnel à connexion affine. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 133–141. (Romanian. Russian and French summaries)

In a space with affine connection, let x^i $(i=1, 2, \cdots, n)$ be a point and X^i a direction through that point. The author considers the problem of invariants under parallel transport of the form $f(x^i, X^i)$. The condition of invariance under parallel transport leads to a complete system of partial differential equations which admits a number of invariants depending upon the number of linearly independent equations in the system. The author then considers the cases $n=2$, 3. For $n=2$, the spaces which admit one invariant are those with absolute parallelism. For $n=3$, the spaces with one invariant are of four types which admit one, two, or three fields of parallel directions. If $n=3$ and the space admits two invariants, it is with absolute parallelism. *R. Blum* (Saskatoon, Sask.)

7526:

Mocanu, Petre. Invariants projectifs des espaces A_n, à connexion affine symétrique, partiellement projectifs, P_n^{n-3} généraux. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 385–400. (Romanian. Russian and French summaries)

A space with affine connection A_n is "partially projective" if a coordinate system can be chosen such that the autoparallel curves lie in linear subspaces of A_n. If these linear subspaces are of dimension k the partially projective space is denoted by P_n^{n-k}. There are two classes of such spaces: The "Kagan" spaces for which the abovementioned linear subspaces pass through a fixed point and the general partially projective spaces for which the linear subspaces containing the autoparallel lines are arbitrary.

The coefficients Γ_{jk}^{i} of the connection of a general P_n^{n-k} satisfy a system of partial differential equations of order $k-1$. This paper is concerned with the establishment of the invariants of this system under an arbitrary projective transformation of the connection, in the case in which $k=3$. For larger values of k the calculations become too complicated.

In addition the author points out that, since the partial differential system which is satisfied by the coefficients of the connection of a Kagan P_n^{n-k-1} is similar (and of the same order) to that of a general P_n^{n-k}, the problem of finding the projective invariants of a Kagan P_n^{n-k} is equivalent to that of finding the projective invariants of a general P_n^{n-k+1}.

R. Blum (Saskatoon, Sask.)

7527:

Haimovici, Adolf. Espaces à métrique angulaire. III. Acad. R. P. Romîne. Fil. Iaşi Stud. Cerc. Şti. Mat. **9** (1958), no. 2, 131–139. (Romanian. Russian and French summaries)

The author has introduced in previous papers [see parts I, II: same Stud. **1** (1951), 157–163; **2** (1951), 66–82; MR **17**, 408] in spaces with affine connection L_n the notion of angle as a function V of a point x^i and two vectors X^i, Y^i satisfying the following conditions: (1) V is homogeneous of degree zero with respect to X^i and also with respect to Y^i. (2) V is invariant under the parallel transport of the vectors X^i and Y^i. (3) V is invariant under an allowable transformation of coordinates. (4) When $n=2$, V satisfies an additive property of the form

$$(*) \qquad V(x^i; X^i; Y^i) + V(x^i; Y^i; Z^i) = V(x^i; X^i; Z^i),$$

and when $n>2$, V has the same value in all subspaces of L_n tangent to X^i and Y^i.

In the present paper this last condition (4) is replaced by the following. (4′) If Z^i is coplanar to X^i and Y^i, i.e., $Z^i=\alpha X^i+\beta Y^i$, where α, β are two arbitrary scalars, then the relation (*) is valid.

Utilising previous methods the author finds four types of spaces L_n which admit the notion of angle satisfying the conditions (1), (2), (3) and (4′). In each of these cases he gives the analytic expression of the angle, and points out that a particular case of the fourth type is that of the Riemannian spaces. *R. Blum* (Saskatoon, Sask.)

7528:

Badea, Mauriciu. Espaces A_3 à connexion affine constante localement euclidiens, et algèbres associées. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 559–590.

A space with affine connection A_n is with "constant connection" if a coordinate system can be found in which the components of the connection Γ_{jk}^{i} are constants. These spaces have been introduced by G. Vrănceanu who has pointed out that to such an A_n there can be associated an algebra of hypercomplex numbers (with n hypercomplex units), which is commutative and associative if the curvature tensor and the torsion tensor of the A_n are zero.

In the present paper the author studies the spaces A_3 with constant affine connection and satisfying Vrănceanu's conditions for which the connection vector $\Gamma_k=\Gamma_{ik}^{i}\neq 0$. In this case the author shows that there exist essentially ten classes of A_3 and their associated algebras, for which he obtains the corresponding canonical forms.

This is done in the following way: First the connection vector Γ_k is reduced to the canonical form $(0, 0, \lambda)$ and then the quadratic form $F^3=\Gamma_{jk}^{3}x^j x^k$ is considered. The following results are obtained. (1) If F^3 is a non-degenerate conic with center, the A_3 can be reduced to two different canonical forms. (2) If F^3 is degenerated into two parallel straight lines, the A_3 can be reduced to three different canonical forms. (3) If F^3 is degenerated into the line at

infinity (taken twice), the A_3 can be reduced to five different canonical forms. (4) In every other case no corresponding A_3 of the type discussed exists.

R. Blum (Saskatoon, Sask.)

7529:

Kanitani, Jôyô. Sur le faisceau canonique d'une variété différentiable. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 153–170.

By means of the holonomy group of a dominant projective connection established in a previous paper [same Mem. **30** (1956/57), 203–216; MR **19**, 1193] for a differentiable manifold of dimension $n \geq 2$, the author gives a geometrical interpretation for a quantity which is expressed in local coordinates by the same expression as that of the canonical pencil of a surface in a three-dimensional projective space.

C. C. Hsiung (Madison, Wis.)

7530:

Teleman, K. A theorem of Borel and Lichnerowicz. Rev. Math. Pures Appl. **3** (1958), 107–115. (Russian)

After first giving some details of the proof of the theorem in question—a Riemann space is locally indecomposible if and only if its local homogeneous holonomy group is irreducible—the author examines some consequences (all local) of this fact. Theorem: If V_n is irreducible and has the same curvature tensor as $\overline{V}_n$, the two metrics differ by a multiplicative constant. This implies a result of Vrănceanu [same Rev. **1** (1956), no. 3, 147–165; MR **20** #6130] asserting the similarity of spaces with the same Christoffel symbols. Spaces for which the curvature tensor is constant (in some coordinate system) are found to be reducible, generalizing a result of Sumitomo [Tensor (N.S.) **5** (1956), 201–204; MR **18**, 66]. The case where this tensor is constant on a system of congruences which are also invariant under a group leads to an expression for the line element in terms of real characters of the group. Finally, two spaces with the same irreducible holonomy group are examined with respect to their consequent conformal equivalence. *L. W. Green* (Minneapolis, Minn.)

7531:

Vidal Abascal, E. A generalization of integral invariants. Proc. Amer. Math. Soc. **10** (1959), 721–727.

Let

$$(*) \qquad \omega_1 = 0, \omega_2 = 0, \cdots, \omega_h = 0$$

be a completely integrable Pfaffian system in m parameters ($h < m$), and consider the ensemble of integral varieties of the system (*) of $m-h$ dimensions, which fill the representative space R of m dimensions. Let Ω be a class C^2 exterior differential form of degree p in R. In this paper the author studies, with some applications, certain absolute and relative integral invariants $\int \Omega$ for the system (*), and the forms Ω yielding a certain integral relation of absolute or relative invariance of the system (*).

C. C. Hsiung (Madison, Wis.)

7532:

Uhlmann, Armin. Gemischte Differentiale und die Theorie von Hodge auf nicht notwendig orientierbaren Mannigfaltigkeiten. Wiss. Z. Friedrich-Schiller-Univ. Jena **8** (1958/59), 31–35.

The author generalizes the theorem of Hodge to non-orientable manifolds by use of mixed differentials. A mixed differential is an assignment of an ordered pair of differential forms to each coordinate system, such that if (α_1, α_2) corresponds to $\{x^1, \cdots, x^n\}$ and (β_1, β_2) corresponds to $\{y^1, \cdots, y^n\}$, then, on the overlap,

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2 \quad \text{if} \quad \frac{\partial(x^1, \cdots, x^n)}{\partial(y^1, \cdots, y^n)} > 0$$

and

$$\alpha_1 = \beta_2, \quad \alpha_2 = \beta_1 \quad \text{if} \quad \frac{\partial(x^1, \cdots, x^n)}{\partial(y^1, \cdots, y^n)} < 0.$$

J. J. Kohn (Waltham, Mass.)

7533:

Sommer, Friedrich. Komplex-analytische Blätterung reeler Hyperflächen im C^n. Math. Ann. **137** (1959), 392–411.

This paper continues the study of various conditions for the existence of decompositions of a real submanifold of the space C^n of complex n-tuples into leaves which are complex analytic submanifolds of C^n, as begun in an earlier paper [same Ann. **136** (1958), 111–113; MR **21** #730]. In this case the only submanifolds considered are hypersurfaces, i.e., of (real) dimension $2n-1$, and hence more precise and more useful criteria are obtained. Let $x = (x_1, \cdots, x_n)$ denote an n-tuple of complex numbers; then a hypersurface is given by $\phi(x, \bar{x}) = 0$. Denote by H the Hermitian form

$$H = \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial x_j \partial \bar{x}_i} u_i \bar{u}_j$$

subject to

$$\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} u_i = 0, \quad \sum_{i=1}^{n} \frac{\partial \phi}{\partial \bar{x}_i} \bar{u}_i = 0.$$

The author obtains the following sufficient condition: let $d'\phi = \sum(\partial\phi/\partial x_j)dx^j$ and $d''\phi = \sum(\partial\phi/\partial \bar{x}_j)d\bar{x}^j$; if

$$d'\phi \wedge d''\phi \wedge (d'd''\phi)^r$$

vanishes for $r = k$ but not for $r = k-1$, then the hypersurface M is decomposable as a $(2k-1)$-parameter family of $(n-k)$-dimensional complex analytic manifolds. It is shown by a simple example that this condition is not necessary in general. However, if the form H is semi-definite, it is necessary also. The integer k is in fact the rank of H augmented by one, and letting i denote its index, the author obtains the more precise result: the hypersurface is decomposable into a $(2s-1)$-parameter family of complex analytic leaves of complex dimension $n-s$, where $n-k \leq n-s \leq n-\frac{1}{2}(k+|i|+1)$.

W. M. Boothby (St. Louis, Mo.)

7534:

Elianu, Jean. Fonctions complexes et groupes continus. Bull. Sci. Math. (2) **81** (1957), 134–144.

É. Cartan [Leçons sur le géométrie des espaces de Riemann, 2ième éd., Gauthier-Villars, Paris, 1946, p. 76; MR 8, 602] has shown that the theory of analytic functions of a complex variable leads to regular Euclidean metrics. Guided by this idea, the author associates a Riemannian space V_2 with a regular metric $(ds)^2 = (ds^1)^2 + (ds^2)^2 = (P^2+Q^2)(dx^2+dy^2)$ to every complex function $f(z) = P(x, y) + iQ(x, y)$ without zero in a certain domain R, where $ds^1 = Pdx - Qdy$, $ds^2 = Qdx + Pdy$, and applies a

theorem of G. Vrănceanu [C. R. Acad. Sci. Paris **189** (1929), 386–389] on congruences (ds^1, ds^2) and thus is led to a simply transitive continuous group of rigid displacements

$$X_1\varphi = (P\varphi_x - Q\varphi_y)/(P^2+Q^2),$$
$$X_2\varphi = (Q\varphi_x + P\Phi_y)/(P^2+Q^2).$$

When this group is commutative, the associated Riemannian space is locally Euclidean and the complex function is analytic. In the contrary case, the associated space is either that of constant negative curvature ($n=1$) or a Cartan symmetric space ($n>1$) and thus we are led to a class of non-analytic complex functions. Such complex functions are studied. The bi-linear covariants of the forms ds^1, ds^2 are $w_{12}{}^1 ds^1 \wedge ds^2$, $w_{12}{}^2 ds^1 \wedge ds^2$, where

$$w_{12}{}^1 = (Q_x+P_y)/(P^2+Q^2), \qquad w_{12}{}^2 = (Q_y-P_x)/(P^2+Q^2).$$

The connection parameters $\Gamma_{bc}{}^a$ taken on the congruences ds^1 and ds^2 are the coefficients of rotation of Ricci on ds^1, ds^2.

Main results: (i) A complex function without zero in R and with constant coefficients of rotation of Ricci is analytic when and only when V_2 is locally Euclidean. (ii) The areal derivative [Pompeiu, Rend. Circ. Mat. Palermo **33** (1912), 108–113] is

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\sqrt{(-\gamma_{212}{}^1)}|f(z)|^2 e^{i\psi},$$

($\gamma_{212}{}^1 = \partial\Gamma_{21}{}^1/\partial s^2 - \partial\Gamma_{22}{}^1/\partial s^1) - (\Gamma_{21}{}^1)^2 - (\Gamma_{22}{}^1)^2$, $\cos\psi = \Gamma_{22}{}^1/\sqrt{(-\gamma_{212}{}^1)}$, $\sin\psi = \Gamma_{11}{}^2/\sqrt{(-\gamma_{212}{}^1)}$). (iii) The first integral formula:

$$\int Cf(z)dz - i\sqrt{(-\gamma_{212}{}^1)}e^{i\psi}\iint_D |f(\zeta)|^2 d\xi d\eta = 0$$

($\zeta=\xi+i\eta$ in D, C the boundary of D). (iv) The second integral formula:

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a} - \frac{1}{2\pi}\sqrt{(-\gamma_{212}{}^1)}e^{i\psi}\iint_D \frac{|f(\zeta)|^2}{\zeta-a} d\xi d\eta.$$

(v) Extension of the above results to the case of a function of n complex variables.

{Remark: The congruences ds^1, ds^2 should be compared to the reviewer's H-geodesic curves [e.g., Proc. Japan Acad. **34** (1958), 471–476; Yokohama Math. J. **6** (1958), 89–176; MR **20** #7290; **21** #5994].}

T. Takasu (Yokohama)

7535:

Nagai, Tamao. On some properties of congruences of curves in Riemannian spaces admitting simply transitive groups. Tensor (N.S.) **9** (1959), 104–112.

The author studies the relationship between the structure of a simply transitive group G_n in a V_n and properties of orthogonal congruences of curves in V_n, describing some of the results in terms of the angular spread vector and distantial spread vector introduced by W. C. Graustein [Trans. Amer. Math. Soc. **34** (1932), 557–593; **36** (1934), 542–585] and generalized by R. M. Peters [Amer. J. Math. **57** (1935), 103–111; **59** (1937), 567–574]. Let $\xi_{(a)}{}^i$ be a set of n linearly independent vectors and let G_n be generated by the infinitesimal transformations $X_a f = \xi_{(a)}{}^i \partial f/\partial x^i$. Then, of ten results obtained, three are the following. (a) A necessary and sufficient condition that $X_p f$ generate an Abelian subgroup is that $\mu_{(pq)}{}^c = \mu_{(qp)}{}^c$ for the angular spread vector $\mu_{(pq)}{}^c$ ($p, q = 1, \cdots, m$; $c = 1, \cdots, n$); alternatively, the distantial spread vector $\nu_{(pq)}{}^c = 0$. (b) The order of the derived group of G_n is equal to the number of linearly independent distantial spread vectors of the corresponding orthogonal families of curves in V_n. (c) For the linear connection $\Lambda_{jk}{}^i = \xi_{(a)}{}^i \partial\xi_k{}^{(a)}/\partial x^j$ in V_n, the components of the angular spread vector vanish and the components of the distantial spread vector and the torsion vector are the same as the constants of structure of G_n.

A. Schwartz (New York, N.Y.)

7536:

Voss, K. Über geschlossene Weingartensche Flächen. Math. Ann. **138** (1959), 42–54.

A surface in the 3-dimensional Euclidean space is called a W-surface if its mean curvature H and Gaussian curvature K are functionally dependent. Let F be an analytic W-surface which does not lie in any plane or sphere. For each umbilic p_0 of F, Hopf [Math. Nachr. **4** (1951), 232–249; MR **12**, 634] showed that

$$\lambda = \lim_{p\to p_0} |H-H_0|/(H^2-K)^{1/2}$$

always exists, where $H=H(p)$, $K=K(p)$ and $H_0=H(p_0)$. In this paper the following are proved. (i) Locally, the set of umbilics with $\lambda=1$ forms a regular curve if not empty. (ii) The field of lines of curvature can be extended continuously to umbilics with $\lambda=1$. (iii) If F has an umbilic with $\lambda>1$, then F is a part of a surface of revolution. (iv) A compact analytic W-surface of genus zero is a surface of revolution.

H. C. Wang (Evanston, Ill.)

7537:

Kurita, Minoru. A note on umbilics of a closed surface. Nagoya Math. J. **15** (1959), 219–223.

Let M be a 2-dimensional orientable Riemann manifold. In a coordinate neighborhood U, let e_1, e_2 be a rectangular frame such that $ds^2 = \omega_1{}^2 + \omega_2{}^2$. Let ω_{12} be such that $d\omega_1 = \omega_2 \wedge \omega_{12}$, $d\omega_2 = \omega_1 \wedge \omega_{12}$, $\omega_{12} = -\omega_{21}$, and so $d\omega_{12} = K\omega_1 \wedge \omega_2$. Let X be a direction field on M with isolated singularities. In U, let X be given by (l_1, l_2) with respect to the rectangular frame. In terms of the covariant differentials Dl_1, Dl_2, put

$$\tau = (l_1 Dl_2 - l_2 Dl_1)(l_1{}^2 + l_2{}^2)^{-1},$$

so that $\tau = d \arctan(l_2/l_1) + \omega_{12}$. This relation and Gauss-Bonnet are used to derive the standard result that if M is compact, χ is its Euler-Poincaré characteristic and X has a finite number of singularities, then χ is the sum of the indices of these singularities. If D is a simply connected domain on a surface in E_3 and the boundary of D consists of p arcs on lines of curvature with inner angles equal to $\frac{1}{2}\pi$, then the sum of the indices of the umbilics in D is $2-\frac{1}{2}p$.

P. Hartman (Baltimore, Md.)

7538:

Mihăileanu, N. Sur l'immersion d'un V_m dans un V_n. Com. Acad. R. P. Romîne **9** (1959), 101–103. (Romanian. Russian and French summaries)

Let (1) $ds^2 = g_{ij}dx^i dx^j$ be a Riemannian space V_n and $x^i = x^i(u^1, u^2, \cdots, u^m)$ an allowable transformation of its

coordinates. Then the Christoffel symbols of the second kind Γ_{jk}^{h} with respect to the u^i are given by the formula:

$$\Gamma_{jk}^{h} = \frac{\left(\frac{\partial x^i}{\partial u^1}, \cdots, \frac{\partial x^i}{\partial u^{h-1}}, \frac{D}{\partial u^j}\frac{\partial x^i}{\partial u^k}, \frac{\partial x^i}{\partial u^{h+1}}, \cdots, \frac{\partial x^i}{\partial u^n}\right)}{\left(\frac{\partial x^i}{\partial u^1}, \cdots, \frac{\partial x^i}{\partial u^n}\right)}$$

where D is the symbol of absolute differentiation with respect to (1). This is a generalization of a formula found by the same author for the Christoffel symbols of the second kind of a Euclidean space [see Trav. Conf. Géom. Diff. 1955, pp. 161–167, Edit. Acad. R. P. Romîne, Timişoara; Com. Acad. R. P. Romîne **6** (1956), 757–762; MR **19**, 447, 448].

With the help of this formula it is shown that, when a V_m is imbedded in the V_n according to a method given by Vrănceanu, then the coefficients in the formulae of Gauss and Weingarten are really the Christoffel symbols of the second kind of the imbedding space V_n calculated on the imbedded space V_m.

Similarly the law of transformation of the curvature tensor of V_n with respect to the two sets of variables x^i and u^i becomes, when calculated on V_m, the fundamental equations (of Gauss, Codazzi and Ricci-Kuehne) of the V_m in the V_n. *R. Blum* (Saskatoon, Sask.)

7539:

Stanciu, P. Submersion d'une variété riemannienne dans un espace non euclidien. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 127–131 (1959). (Romanian. French and Russian summaries)

The metric of a non-euclidean space S_n, written in terms of homogeneous coordinates, can be interpreted as the metric of a euclidean space of one dimension more. This observation by N. Mihaileanu is used in this paper to establish (1) the equations of Gauss and Codazzi for a V_{n-1} imbedded in an S_n, and (2) the equations of Gauss, Codazzi and Ricci-Kuehne for a V_m imbedded in an S_n.
R. Blum (Saskatoon, Sask.)

7540:

Hsiao, E. K. Notes on differential geometry. Amer. Math. Monthly **66** (1959), 898–899.

Two theorems on surfaces of constant width are proved, pointing out relations between certain curvatures at opposite points. *L. W. Green* (Minneapolis, Minn.)

7541:

Busemann, Herbert. Minkowski's and related problems for convex surfaces with boundaries. Michigan Math. J. **6** (1959), 259–266.

The following theorems are proved [for the nomenclature see the author's book *Convex surfaces*, Interscience Publishers, New York-London, 1958; MR **21** #3900]. (1) Let K, K' be convex hypersurfaces in E^n homeomorphic to E^{n-1} and with the same boundary B homeomorphic to S^{n-2}, such that the supporting planes of K and K' do not contain points of B. If K and K' have the same spherical image and the same mth area function (m fixed), then $K=K'$. (2) If two spatial caps K, K' have the same mth area function for all sets X on the unit sphere Σ of the spherical images of K and K', then K' originates from K by translation. (3) Let $a(X)$ be a non-negative completely additive set function defined on all Borel sets X on Σ satisfying conditions: (a) $a(X)=0$ when X lies in $x_n<0$; (b) $\int_\Sigma u_j a(d\Sigma_u)=0$ for $j<n$; (c) $a(G_u)<a(\Sigma)$ for all u (where G_u denotes the intersection of Σ with $u\cdot x=0$); then there exists, up to translations, exactly one spatial convex cap K with $a_{n-1}^0(K, X)=a(X)$ for all Borel sets $X\subset\Sigma$ and its boundary lies in a plane $x_n=$ constant. The author points out that these and other results of the theory of convex bodies hold good for surfaces with boundaries, just because the theory of convex bodies is not restricted to smooth surfaces. *L. A. Santaló* (Buenos Aires)

7542:

Pogorelov, A. V. The rigidity of general convex surfaces. Dokl. Akad. Nauk SSSR **128** (1959), 475–477. (Russian)

With no proofs or only the sketchiest indications, various important results on the rigidity of convex surfaces are stated. Let F, given by $x=x(u, v)$, be a not necessarily complete convex surface in E^3, and $\tau(u, v)$ a continuous vector field on F. Then $x(u, v)+\varepsilon\tau(u, v)$ defines a deformation F_ε of F which is an infinitesimal isometry if any curve γ on F and the corresponding curve γ_ε on F_ε satisfy (length γ − length γ_ε)/$\varepsilon\to 0$ for $\varepsilon\to 0$. The vector field τ corresponds to a rigid motion of F if $\tau=a\times x+b$, and is called trivial on any part of F where it has this form. Under differentiability hypotheses on F and τ it has been known for a long time that closed convex surfaces with positive curvature are rigid, i.e., that the only τ yielding infinitesimal isometries are trivial on all of F. This result holds for any closed convex F without plane pieces (obviously necessary!) and arbitrary continuous τ. If F has plane pieces then τ will be trivial outside of the plane pieces. If τ defines an infinitesimal isometry of a regular (not necessarily complete) convex surface of positive Gauss curvature and of class $C^{n+\alpha}$, $n\geq 2$, $\alpha>0$, then τ is of class $C^{n+\alpha'}$, $0<\alpha'<\alpha$, and τ is analytic with F. These results hold also for spaces of constant curvature (instead of E^3).

The author indicates how these results lead to a considerable reduction in his proof of the theorem that two intrinsically isometric closed convex surfaces are congruent. *H. Busemann* (Los Angeles, Calif.)

7543:

Sun', Hè-Sèn. Some problems bearing on infinitesimal flexures of surfaces. Dokl. Akad. Nauk SSSR **122** (1958), 559–561. (Russian)

Continuing his study of infinitesimal rigidity of surfaces [same Dokl. **116** (1957), 758–761; MR **20** #288], the author applies results of B. Boyarskiĭ [same Dokl. **102** (1955), 661–664; 871–874; MR **17**, 157] on the number of independent solutions of the deformation equations. Let L be the closed boundary curve of the surface S, θ the angle the principal normal of L makes with n, the normal to S, τ the angle between the binormal of L and the constraint normal ν, and U the deformation vector. Typical theorems are the following. Theorem 1: If S is an ovaloid with noncircular plane boundary L, and, on L, $U\cdot\nu=0$, $\tau\neq 0$ or $\pi/2$, and $\sin(\theta-\tau)\neq 0$, then S admits three or more nontrivial infinitesimal deformations. Theorem 2: S is rigid under deformations with constraint $U\cdot n=0$ on L if and

only if L is a circle and θ is constant. Boyarskiĭ's results also give information when S has several openings. Specific applications are made to the case of a surface of revolution containing a piece of a cylinder.

L. W. Green (Minneapolis, Minn.)

7544:

Kurita, Minoru. On the volume in homogeneous spaces. Nagoya Math. J. **15** (1959), 201–217.

Let M be the homogeneous space G/H, where G is a Lie group of dimension r and H a closed connected subgroup of dimension $r-n$. Let $x_1, x_2, \cdots, x_n=u$ be cartesian coordinates of the euclidean space E_n and consider a domain D on the hyperplane $u=0$. Let ψ be a 1-1 mapping of D into M such that $\psi(D)$ is an $(-n1)$-dimensional submanifold of M. By a 1-parametric motion represented by $S_t \in G$ $(t=t(u),\ 0 \leq u \leq 1)$, $\psi(D)$ generates a set in M. Theorem: The algebraic volume (appropriately defined) generated by $\psi(D)$ has the form $V=\sum_{p=1}^{r} X_p Y_p$, where X_p are quantities determined by $\psi(D)$ and Y_p are quantities determined by the motion. If $\psi(D)$ is transformed into itself transitively by a connected closed subgroup of G, the form of V admits some simplifications. The case of the euclidean space with points as its elements gives a known theorem of Königs [J. Math. Pures Appl. (4) **5** (1889), 321–343] and Hadamard [Bull. Sci. Math. **26** (1898), 264–265]; the cases of euclidean space with either hyperplanes or straight lines as its elements give new results which are considered in detail.

L. A. Santaló (Buenos Aires)

7545:

Maebashi, Toshiyuki. A weakly osculating Riemann space of the Finsler space and its application to a theory of subspaces in the Finsler space. Tensor (N.S.) **9** (1959), 62–72.

The "osculating" Riemannian space of a given Finsler space is obtained by giving previously assigned values to the directional arguments x'^k of the metric tensor $g_{ij}(x^k, x'^k)$ of the Finsler space, as for instance by means of the tangent vectors of a geodesic [O. Varga, Monatsh. Math. Phys. **50** (1941), 165–175; MR **5**, 218]. This concept was generalised by A. Moór to the case of affinely connected spaces of line elements [Ann. of Math. (2) **56** (1952), 397–403; MR **14**, 321]. A similar idea is introduced in the present paper, where, however, the coincidence refers not to a curve but only to a certain vector field (hence the term "weakly" osculating space). This allows the author to define a connection in the Finsler space (identical with that of Cartan). This concept is applied to the theory of subspaces. The author concludes that the intrinsic and induced connections of the subspace are identical if and only if the space is Riemannian.

H. Rund (Durban)

7546:

Tonooka, Keinosuke. On three and four dimensional Finsler spaces with the fundamental form $(\sqrt[3]{a_{\alpha\beta\gamma}x'^{\alpha}x'^{\beta}x'^{\gamma}})$. Tensor (N.S.) **9** (1959), 209–216.

The principal purpose is to introduce the affine connection in the Finsler space $F_n^{(3)}$ $(n=3, 4)$ in which the metric is given by $ds^3=a_{\alpha\beta\gamma}dx^{\alpha}dx^{\beta}dx^{\gamma}$. The affine connection $\Gamma_{\mu\nu}^{\lambda}(x)$ in $F_3^{(3)}$ was studied by the author in a previous paper [Tensor (N.S.) **6** (1956), 60–68; MR **18**, 232] of which an abridgment is stated at the beginning of the present paper. Then the necessary and sufficient conditions are found for the connection such that $\nabla_\lambda a_{\alpha\beta\gamma}=0$ to exist and for the space to be flat. In order that the auto-parallel curves for the affine connection $\Gamma_{\mu\nu}^{\lambda}$ be the same as the extremal curves, the condition $\nabla_\lambda a_{\alpha\beta\gamma}=0$ is necessary and sufficient. The conformal theories are also discussed. Last, the similar theory in $F_4^{(3)}$ is studied.

A. Kawaguchi (Sapporo)

7547:

Nasu, Yasuo. On similarities in a Finsler space. Tensor (N.S.) **9** (1959), 175–189.

A similarity of a, not necessarily symmetric, metric space R with distance xy is a mapping $x \to x'$ of R on itself such that $x'y'=k \cdot xy$, $k>0$, $k \neq 1$. The reviewer proved [Tôhoku Math. J. (2) **9** (1957), 56–67; MR **20** #2772] that a G-space which possesses a similarity need not be Minkowskian, but is Minkowskian under a certain differentiability hypothesis. The author considers a manifold of class C^1 with a function $F(x, \xi)$ defined on all contravariant vectors $(x, \xi)=(x_1, \cdots, x_n, \xi^1, \cdots, \xi^n)$ continuous in the $2n$ variables, positive for $\xi \neq 0$, and with $F(x, k\xi)=kF(x, \xi)$ for $k \geq 0$. The distance xy (in general $xy \neq yx$) is defined as the greatest lower bound of the F-lengths of all curves of class C^2 from x to y. The metric space R thus defined is assumed to be complete with respect to xy. It is known that the length of a curve of class C^1 with respect to xy equals the $\overline{F}$-length, where $\overline{F}(x, \xi)=1$ is for each x the boundary of the convex closure (in ξ-space) of $F(x, \xi)=1$ [H. Busemann and W. Mayer, Trans. Amer. Math. Soc. **49** (1941), 173–198; MR **2**, 225]. It is proved that R is Minkowskian if it possesses a similarity.

H. Busemann (Los Angeles, Calif.)

7548:

Eliopoulos, Hermes A. Sur la définition de la courbure totale d'une hypersurface plongée dans un espace de Finsler localement minkowskien. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 205–214.

The parametric equations $x^i=x^i(u^\alpha)$ $(i, j=1, \cdots, n;\ \alpha, \beta=1, \cdots, n-1)$, represent a hypersurface F_{n-1} of a Finsler space F_n. The metric tensor of F_{n-1}, defined by $g_{\alpha\beta}=g_{ij}(\partial x^i/\partial u^\alpha)(\partial x^j/\partial u^\beta)$ in terms of the metric tensor g_{ij} of F_n, defines a unit indicatrix of volume B. A further space f_{n-1} is defined by $n^i=n^i(u^\alpha)$, n^i being the unit normal of F_{n-1}, with a "metric tensor" defined by $N_{\alpha\beta}=g_{ij}n^i_{;\alpha}n^j_{;\beta}$, the semi-colons denoting covariant differentiation. {Reviewer's note: f_{n-1} is not necessarily a Finsler space, as the $N_{\alpha\beta}$ do not, in general, represent second derivatives of a scalar function with respect to directional arguments.} This tensor defines a second unit indicatrix of volume b —presumably both B and b are to be defined with respect to the same associated euclidean space [Busemann, Comment. Math. Helv. **24** (1950), 156–187; MR **12**, 527]. It is suggested that the total curvature of F_{n-1} at any point be defined by the value of B/b at that point, and it is shown that this value is not in general equivalent to Ω/γ as in Riemannian geometry, where Ω and γ denote the determinants of the second and first fundamental forms of F_{n-1} respectively. A relation between this total curvature and the invariants resulting from the curvature tensor of F_{n-1} is sought for the case $n=3$.

H. Rund (Durban)

PROBABILITY

See also 7397, 7622.

7549:

Heuer, Gerald A. Estimation in a certain probability problem. Amer. Math. Monthly **66** (1959), 704–706.

Let an experiment with N equally likely outcomes be performed K times ($K<N$). The probability, t, that at least one of the outcomes occurs twice is

$$t = 1-\frac{N!}{(N-K)!N^K}.$$

For given N and K, t may be evaluated directly, or approximated with the aid of Stirling's formula. However, the problem becomes more difficult if one wishes to solve for K as a function of t and N. It is shown that, for $0<t<1$, K is given asymptotically by

$$N^{1/2}[-2\log(1-t)]^{1/2}.$$

This approximation is good even for small N.

M. Dresher (Pacific Palisades, Calif.)

7550:

Sarmanov, O. V.; and Zaharov, V. K. Spectra of enlarged stochastic matrices. Dokl. Akad. Nauk SSSR **128** (1959), 243–245. (Russian)

Let $\{A_i\}$ and $\{B_i\}$, $i=1,\cdots,n$, be dependent events with $p_{ij}=P(A_i\cap B_i)$. Let $p_i=\sum_{j=1}^{n}p_{ij}$. Then $P=\{p_{ij}/p_i\}$ is a stochastic matrix. Let $1, 1/\lambda_1, \cdots, 1/\lambda_{n-1}$ be the proper values of P. Consider also the events

$$\{A_1\cup A_2, A_3, \cdots, A_n\}$$

and

$$\{B_1\cup B_2, B_3, \cdots, B_n\}.$$

Let P' and $1, 1/\lambda_1', 1/\lambda_2', \cdots, 1/\lambda_{n-2}'$ be the corresponding quantities for these events. Then the authors show that $1/\lambda_i'=\sum_{k=i}^{n-1}a_{ik}^2/\lambda_k$, where $\sum_{k=i}^{n-1}a_{ik}^2=1$. A number of corollaries follow from more specific information given about the values of a_{ik}.

J. L. Snell (Hanover, N.H.)

7551:

Gilbert, E. N. Random graphs. Ann. Math. Statist. **30** (1959), 1141–1144.

The random graphs in question are produced as follows. Consider the complete graph with N labeled points and $N(N-1)/2$ lines; then the random graph is constructed by making a random choice, with probability p, for each of the lines independently. For such random graphs, the author determines the probability, $P_N\equiv P_N(p)$, that the graph is connected and also the probability, $R_N\equiv R_N(p)$, that two specific points are connected. For the former an explicit formula as well as a recurrence are given, but the latter is determined only in terms of the former. Main attention goes to the asymptotic relations: $P_N\sim 1-Nq^{N-1}$, $R_N\sim 1-2q^{N-1}$, $q=1-p$.

J. Riordan (New York, N.Y.)

7552:

Andrushkiw, Joseph W. The probability that the roots of a real quadratic equation lie inside or on the circumference of the unit circle in complex plane. Math. Mag. **32** (1958/59), 123–128.

The probability in question is defined as the volume of the set of points in the cube $-\frac{1}{2}\le x, w, y\le\frac{1}{2}$ for which the roots of $xz^2+wz+y=0$ lie in $|z|\le 1$. Direct geometrical arguments are used to evaluate this probability ($=7/24$). Probabilities of some similar events are computed en route.

A. Dvoretzky (Jerusalem)

7553:

Ghurye, S. G.; and Wallace, David L. A convolutive class of monotone likelihood ratio families. Ann. Math. Statist. **30** (1959), 1158–1164.

Let an MLR family of densities mean one with monotone likelihood ratios, to be precise, $f(x,\theta)f(x',\theta')\ge f(x,\theta')f(x',\theta)$ for $x<x'$, $\theta<\theta'$. The convolution of two MLR families of densities is MLR if (but not only if), for each value of the parameter, the translation families given by the two densities are also MLR. The two-fold convolution of a single MLR family of discrete densities on the points 0, 1, 2 is MLR. The n-fold convolution of a single MLR family of discrete densities on the vertices of the unit square (in 2 dimensions) or of the unit simplex (in any number of dimensions) is MLR (where, in the definition above, $x<x'$ means no coordinate of $x-x'$ is positive).

J. W. Pratt (Cambridge, Mass.)

7554:

Ruegg, Alain. Intégration d'un ensemble de fonctions caractéristiques par rapport à un paramètre. C. R. Acad. Sci. Paris **249** (1959), 494–495.

Given a 1-dimensional distribution function ψ depending on a stochastic parameter, the average of its transform $\varphi=\int e^{itx}d\psi$ is the transform of its average, i.e., $E(\varphi)=\int e^{itx}dE(\psi)$.

H. P. McKean, Jr. (Cambridge, Mass.)

7555:

Zolotarev, V. M. Mellin-Stieltjes transformations in probability theory. Teor. Veroyatnost. i Primenen. **2** (1957), 444–469. (Russian. English summary)

The author considers the application of Mellin-Stieltjes transforms, defined by $F(s)=\int_0^\infty x^s dF(x)$, to the solution of problems involving products and ratios of random variables. In the first part the relationship between the Mellin-Stieltjes transform, the Laplace-Stieltjes transform and the characteristic function of a given frequency distribution is discussed, it being assumed that the distributions considered are continuous at the origin. In the next part, the concepts of "cuts" and equivalence of random variables are used systematically to establish precise and limiting relations for stable distributions; many known relations are included as special cases. Finally, the behaviour of stable distributions near critical points is considered.

I. N. Sneddon (Glasgow)

7556:

Laha, R. G. On the laws of Cauchy and Gauss. Ann. Math. Statist. **30** (1959), 1165–1174.

Let X and Y be independent random variables with common distribution function F such that X/Y obeys the symmetric Cauchy law. (It is known that this is the case when F is normal with zero mean, but the converse is not true.) The author shows that under the above assumptions: I: F is symmetric, has an unbounded range and a

continuous positive probability density f with

$$\int_0^\infty f(x)f(ux)x dx = c/(1+u^2)$$

for all u, and some constant c; II: If, moreover, F has finite moments of all orders and (i) $\varphi(z) = E \exp\{iz \log |X|\}$ has no zeros in its region of regularity, then F is normal. The author then constructs a non-normal F with all required properties except (i).

M. Loève (Berkeley, Calif.)

7557:

Kemeny, John G. Generalized random variables. Pacific J. Math. **9** (1959), 1179–1189.

Let T be a metric space whose closed balls are compact, and let R be a countable subset of T. For any probability distribution P on R, let $V_P(t)$ be the expected square of the distance from a point of R to the point t. It is supposed that V_P is not identically ∞. It is shown that there is a non-empty compact subset M_P of T minimizing V_P. The members of M_P are the 'means' of the distribution; the minimum value v_P is the 'variance'. Fréchet [Ann. Inst. H. Poincaré **10** (1948), 215–310; MR **10**, 311] has also used these definitions. If $x_1, x_2, \cdots$ are mutually independent, with a common distribution Q on R, the empirical distribution of the first n has a set M_n of means which converges as $n \to \infty$ to M_Q with probability 1. (Convergence means that M_n finally lies in any neighborhood of M_Q.) Moreover the variance of the nth empirical distribution converges to that of Q with probability 1. More generally, if T is any topological space and V_P is a function with suitable properties, the preceding results remain valid.

J. L. Doob (Urbana, Ill.)

7558:

Hanš, Otto. Measurability of extensions of continuous random transforms. Ann. Math. Statist. **30** (1959), 1152–1157.

If $(\Omega, \mathscr{A})$ and $(\mathfrak{X}, \mathscr{B})$ are two measurable spaces, $\mathfrak{Y}$ an arbitrary non-empty set, and T a mapping of $\Omega \times \mathfrak{Y}$ into $\mathfrak{X}$ such that $\{\{\omega : T(\omega, y) \in B\} : y \in \mathfrak{Y}, B \in \mathscr{B}\} \subset \mathscr{A}$, then T is said to be a random transformation of $\Omega \times \mathfrak{Y}$ into $\mathfrak{X}$. In this paper the author states and proves for random transformations probabilistic analogues of the well-known extension theorems of Tietze and Hahn-Banach.

A. T. Bharucha-Reid (Eugene, Ore.)

7559:

Trotter, H. F. An elementary proof of the central limit theorem. Arch. Math. **10** (1959), 226–234.

The characteristic functions of the standard proof are replaced by linear transformations of continuous functions. The exposition is detailed and lucid enough to be used in a first course in probability.

G. A. Hunt (Princeton, N.J.)

7560:

Schnell, Edit. On a conditional limiting distribution theorem. Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 3–10. (Hungarian and Russian summaries)

Let ξ_n $(n = 1, 2, \cdots)$ be independent and identically distributed random variables having mean 0 and variance 1. Let ϕ denote the common characteristic function of variables ξ_i and let $\limsup_{|t| \to \infty} |\phi(t)| < 1$. Then

$$\lim_{n \to \infty} P\{a_1 \leq \sum_{i=1}^{n} \xi_i < b_1 | a_2 \leq \sum_{i=1}^{n} \xi_i < b_2\} = \frac{b_1 - a_1}{b_2 - a_2}$$
$$(a_2 \leq a_1 < b_1 < b_2).$$

S. Sherman (Philadelphia, Pa.)

7561:

Fisz, H. A central limit theorem for stochastic processes with independent increments. Studia Math. **18** (1959), 223–227.

The author proves the following result. Let

$$\{Y_k(t), 0 \leq t \leq a\} \quad (k = 1, 2, \cdots),$$

where $0 < a < \infty$, be a sequence of real, separable, centered, independent and equally distributed stochastic processes with independent increments and with no fixed points of discontinuity and let the realizations of $Y_k(t)$ be continuous from the right (at $t = a$ from the left). Suppose the variance of $Y_k(a)$ is finite. Then the central limit theorem holds. (This last statement is spelled out precisely in the paper.)

J. Wolfowitz (Ithaca, N.Y.)

7562:

Cameron, R. H.; and Donsker, M. D. Inversion formulae for characteristic functionals of stochastic processes. Ann. of Math. (2) **69** (1959), 15–36.

Soit $\{x_t\}_{t \in [0,1]}$ un processus stochastique dont presque toutes les fonctions $x(t, \cdot)$ sont nulles pour $t = 0$ et satisfont à la condition de Hölder avec $\alpha > 1/2$ et soit $\{p_t\}$ le processus de Wiener. La fonctionnelle caractéristique de x_t est définie, dans des conditions plus générales, par la formule $\Phi(p) = E\{\exp i\int_0^1 x(t)dp(t)\}$.

Les auteurs démontrent la formule

$$E_x\{G[x]\} = \lim_{\lambda \to \infty} 2^{-1/2} \exp(\lambda f(\lambda)/2)$$
$$\times E_a{}^w E_p{}^w\{\exp[-i\lambda f(\lambda)\int_0^1 a(t)dp(t) G[f(\lambda)a]\Phi(\lambda p)]\}$$

pour toute fonctionnelle G, définie et bornée sur l'espace des fonctions continues dans $[0, 1]$ et nulles pour $t = 0$, continue dans la topologie uniforme et mesurable Wiener et pour $0 < f(\lambda) = o(\lambda/(\log \lambda)^2)$, $1/f(\lambda) = o(\lambda^{1-1/\alpha})$.

Une formule analogue est démontrée pour les processus "strictement absolument continus" et une autre pour les fonctionnelles G continues dans la topologie hilbertienne.

G. Marinescu (Bucharest)

7563:

Minlos, R. A. Generalized random processes and their extension in measure. Trudy Moskov. Mat. Obšč. **8** (1959), 497–518. (Russian)

The author proves results announced in a previous paper [Dokl. Akad. Nauk SSSR **119** (1958), 439–442; MR **20** #5522], showing how to set up a stochastic process $\{F(\phi), \phi \in E\}$, where the parameter ϕ is a point of a linear space E, $F(\Sigma c_j \phi_j) = \Sigma c_j F(\phi_j)$ and the basic measure space of the process is the conjugate space of E. The space E is countably normed, and a certain continuity condition is imposed. The characteristic functional of the process, defined by $f(\phi) = \mathrm{Av}\{\exp[iF(\phi)]\}$, is positive, definite and continuous, and is the most general positive definite continuous functional on E with $f(0) = 1$. If the process is Gaussian with zero means, the covariance function

determines a symmetric bilinear positive definite continuous functional, and any such functional determines such a Gaussian process. Finally, it is shown that, in a suitable sense, the conditions imposed on E are necessary.

J. L. Doob (Urbana, Ill.)

7564:

Feldman, J. Correction to "Equivalence and perpendicularity of Gaussian processes". Pacific J. Math. **9** (1959), 1295–1296.

The proof of the theorem in the paper referred to in the title [same J. **8** (1958), 699–708; MR **21** #1546] is amended so as to include the non-separable case.

E. Nelson (Princeton, N.J.)

7565:

Chacon, Rafael V. On a question of Paul Lévy. Proc. Amer. Math. Soc. **10** (1959), 460–465.

Lévy [Ann. Sci. École Norm. Sup. (3) **68** (1951), 327–381; MR **13**, 959] has asked whether each non-measurable transition function $p_{ij}(t)$ of a stationary Markov chain must be a transition function of a Markov chain obtained by combining a Markov chain having measurable transition functions with one having a transition matrix of non-measurable functions that assume only the values zero or one. Lévy [op. cit.] has shown that if $\{p_{ij}(t)\}$ $(i, j = 1, 2, \cdots)$ is a transition matrix of measurable transition functions then $\lim_{t\to\infty} p_{ij}(t)$ exists and

$$(*)\quad \lim_{t\to\infty} p_{ij}(t) = \lim_{t\to\infty} p_{jj}(t) \quad \text{or} \quad p_{ij}(t) = 0 \quad (i, j = 1, 2, \cdots).$$

Lévy [op. cit.] has also essentially shown that if the transition functions are measurable then they are strictly positive or identically zero.

The author shows that if each of the transition functions is strictly positive or identically zero then (*) holds without measurability assumptions and without assuming an affirmative answer to Lévy's question. The author proves a second theorem that has a weaker result than (*), with no conditions on the transition functions. These theorems and their corollaries describe the asymptotic behavior of the transition functions to be expected if Lévy's question has an affirmative answer, and in fact do so independently of whether or not it is affirmative.

H. P. Edmundson (Pacific Palisades, Calif.)

7566:

Mandl, Petr. Sur le comportement asymptotique des probabilités dans les ensembles des états d'une chaîne de Markov homogène. Časopis Pěst. Mat. **84** (1959), 140–149. (Russian. Czech and French summaries)

The author studies the limiting behavior of the probability $P(x_n = \omega | T^n)$ that the system of a finite homogeneous Markov chain is, after n steps, in a state ω belonging to the set T of states of that chain, under the condition that it remained in the states of T at every instant between 0 and n inclusive. The set T is called regular if the transition matrix is indecomposable and aperiodic. The characteristic root of the matrix of largest absolute value is called the characteristic root of the set T. The author proves that if T is regular then

$$\lim_{n\to\infty} P(x_n = \omega | T^n)$$

exists, is positive, and is independent of the initial probability distribution. The author then examines the general case where T is decomposable into several classes T_i of communicating states. If the classes T_i are regular, the limit $\lim_{n\to\infty} P(x_n = \omega | T^n)$ exists. Whether the limit is positive or not is shown to depend on the characteristic roots of T_i and the structure of the set of T_i given by the relation $T_i < T_k$ if the probability is positive that the system passes into T_k given that it was in T_i.

H. P. Edmundson (Pacific Palisades, Calif.)

7567:

Lamperti, John; and Suppes, Patrick. Chains of infinite order and their application to learning theory. Pacific J. Math. **9** (1959), 739–754.

Let $I = \{1, 2, \cdots, N\}$, x = a finite sequence with elements in I, $p_i(x)$ be a set of functions defined for all $i \in I$ such that $p_i(x) \geq 0$ and $\sum_i p_i(x) = 1$, and

$$p_i^{(n)}(x) = \sum_{j\in I} p_j(x) p_i^{(n-1)}(j + x),$$

where $j + x$ is the sequence x with j prefixed. Assume that for some j_0, some positive integer n_0, and some $\delta > 0$, $p_{j_0}^{(n_0)}(x) > \delta$ for every x. Let $\varepsilon_m = \sup |p_i(x + x') - p_i(x + x'')|$, where the sup is taken over all $i \in I$, all sequences x' and x'', and all x which contain j_0 at least m times. Assume $\sum_{m=0}^{\infty} \varepsilon_m < \infty$. Theorem: $\lim_{n\to\infty} p_i^{(n)}(x) = \pi_i$ exist, are independent of x, and satisfy $\sum \pi_i = 1$; the convergence is uniform in x. Similar results hold for joint probabilities and moments. These results on chains of infinite order generalize those of W. Doeblin and R. Fortet [Bull. Soc. Math. France **65** (1937), 132–148]. These results are applied to the linear learning models of Bush and Mosteller [*Stochastic models for learning*, Wiley, New York; Chapman & Hall, London, 1955; MR **16**, 1136] and to s-person generalizations to show that under weak conditions the asymptotic moments of the response probabilities all exist and are independent of the initial distribution provided that the past dependence of the reinforcement schedule is finite.

R. D. Luce (Philadelphia, Pa.)

7568:

Girsanov, I. G. Some topologies connected with a Markov process. Dokl. Akad. Nauk SSSR **129** (1959), 488–491. (Russian)

Various topologies intrinsically related to Markov processes are introduced, and their interrelations discussed without proof. For example, let $\bar{H}$ be the class of functions f for which f is right-continuous as a function of the parameter on almost every process path, for every initial distribution. The topology defined by $\bar{H}$ is the weakest for which every f in $\bar{H}$ is continuous.

J. L. Doob (Urbana, Ill.)

7569:

★Masani, P.; and Wiener, N. Non-linear prediction. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 190–212. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

The authors give a general survey of the non-linear prediction theory of strictly stationary stochastic processes. If $\cdots, f_{-1}, f_0, \cdots$ is such a process, the prediction of f_r given f_n for $n \leq 0$ is accepted as usual as the conditional expectation $E\{f_r | \cdots, f_0\}$. To simplify matters it is assumed that f_0 is bounded. The prediction is then the

Hilbert space projection of f_r on the closed linear manifold generated by polynomials in the f_n with $n \leq 0$, and this projection can (theoretically) be calculated in terms of the moments $E\{f_{n_1} \cdots f_{n_s}\}$. *J. L. Doob* (Urbana, Ill.)

7570:

Wever, Franz. Über stationäre Null-Eins-Prozesse mit beschränkten Lücken. Math. Z. **71** (1959), 283–288.

The author carries further a calculation by H. Akaike [Ann. Inst. Statist. Math. Tokyo **8** (1956), 87–94; MR **19**, 327] of the spectral density function $w(\lambda)$ of a certain stationary "gap process" $\{x_n\}$. In this the random variables x_n equal 0 or 1 and the probability of a run of n zeros between a 1 and the next 1 is p_n, where $p_n = 0$ when $n > k$, and the highest common factor of numbers n with $p_n > 0$ is 1. Writing $1/p$ for $p_1 + 2p_2 + \cdots + kp_k$, B_j for $1 - p_1 - \cdots - p_{k-j-1}$ $(j = 0, 1, \cdots, k-2)$, $B_{k-1} = 1$, and $p_0(z)$ for $B_0 + B_1 z + \cdots + B_{k-1} z^{k-1}$, and putting

$$H_{\pm j} = p \sum_{r=0}^{k-2-j} B_r (1 - B_{r+j}) \quad (j = 0, 1, \cdots, k-2),$$

he finds that

$$w(\lambda) = \sum_{j=2-k}^{k-2} H_j z^j / |p_0(z)|^2,$$

where $z = \exp 2\pi i\lambda$. He deduces that if ξ_n is

$$B_0 x_n + \cdots + B_{k-1} x_{n+k-1} - 1$$

then ξ_m and ξ_n are uncorrelated when $|m - n| > k$.
H. P. Mulholland (Exeter)

7571:

Moy, Shu-Teh Chen. Successive recurrence times in a stationary process. Ann. Math. Statist. **30** (1959), 1254–1257.

The author considers consecutive recurrence times of the event $X \in B$ for a stationary process $\{X_n\}$, and derives results concerning the stationarity and ergodicity of the process constituted by these times.
P. Whittle (Cambridge, England)

7572:

Blachman, Nelson; and Proschan, Frank. Optimum search for objects having unknown arrival times. Operations Res. **7** (1959), 625–638.

"Objects arrive in accordance with a Poisson process. Having arrived, the object appears in box i with probability p_i. A single scan of box i costs C_i, takes time t_i, and, if the object is present in box i at the beginning of the scan, will detect it with probability P_i. The resultant gain $g_i(t)$ is a nonincreasing function of t, the delay between arrival and the beginning of the detecting look; $i = 1, 2, \cdots, n$. An asymptotically optimum search procedure is obtained. A number of particular cases of interest are solved." (From the authors' summary)
J. Kiefer (Ithaca, N.Y.)

7573:

Reed, Irving S. On the use of Laguerre polynomials in treating the envelope and phase components of narrow-band Gaussian noise. Trans. I.R.E. **IT-5** (1959), 102–105.

Narrow-band stationary Gaussian noise with a power spectrum centered at angular frequency ω can be represented as $x(t) = R(t) \cos(\omega t + \theta(t))$, where $R(t)$ is the envelope and $\theta(t)$ the phase modulation. Starting with Rice's formula for the joint density $P(R_1, R_2, \theta_1, \theta_2)$ of $R(t_1)$, $R(t_2)$, $\theta(t_1)$, $\theta(t_2)$, the author studies the quantity $\Gamma_{12}^{(m)}(\tau) = E\{G_1(R_1) G_2(R_2) e^{im(\theta_2 - \theta_1)}\}$, where G_1, G_2 are given complex-valued functions of a real variable satisfying certain growth restrictions. He obtains a formula for $\Gamma_{12}^{(m)}(\tau)$ as an infinite series whose terms involve integrals with respect to Laguerre weight functions. In the special example $G_1(R) = R^\alpha$, $G_2(R) = R^\beta$, $\Re\alpha > -2$, $\Re\beta > -2$, the series can be summed in terms of the hypergeometric function. *E. Reich* (Minneapolis, Minn.)

STATISTICS

7574:

Gumbel, E. J. The mth range. J. Math. Pures Appl. (9) **38** (1959), 253–265.

The mth range w_m of a sample of n is the difference between the mth observation from the top, and from the bottom; w_1 is the range proper. The reduced mth range is a certain linear transform R_m of w_m, with coefficients depending on the particular initial distribution. This is assumed to be symmetrical, to range over the whole real axis, and to possess the "exponential type" property $\lim(-f')/f = \lim f/(1-F)$ $(x \to \infty)$. It is shown that the asymptotic distribution of R_m (m fixed, $n \to \infty$) is expressible in terms of the (previously derived) asymptotic distribution of R_1. Moments are calculated.—For m close to $n/2$, the mth range is asymptotically normally distributed, with mean and standard deviation given in the paper. *G. Elfving* (Helsinki)

7575:

Weiler, H. Note on harmonic and geometric means. Austral. J. Statist. **1** (1959), 44–46.

It is shown that if $a > 0$ is the smallest and b is the largest of $x_1, x_2, \cdots, x_n$, whose arithmetic and harmonic means are $\bar{x}$ and H respectively, then $0 \leq (\bar{x} - H)/H \leq (b-a)^2/4ab$, the first equality holding only if the x's have equal values and the second only if half the x's have the value a and the other half the value b. Thus, he remarks, $\bar{x}$ differs little from H if the x's have a small range and all are far removed from zero. Moreover since the geometric mean g of a set of positive x's lies between the arithmetic mean and the harmonic mean, the same inequality holds for $(\bar{x} - g)/g$. *C. C. Craig* (Ann Arbor, Mich.)

7576:

★Anderson, T. W. Some scaling models and estimation procedures in the latent class model. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 9–38. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

The author considers models with observed variables x_i for each individual and a latent variable y (these variables need not be real numbers). He assumes the x's are conditionally independent given y. He mentions several models of this type, and goes into some detail about factor analysis and discusses the estimation problem for the latent class model in considerable detail.
H. Rubin (East Lansing, Mich.)

7577:

Hall, Wm. Jackson. The most-economical character of some Bechhofer and Sobel decision rules. Ann. Math. Statist. **30** (1959), 964–969.

Given m independent random samples of equal size, one from each of m normal distributions with a common known variance, it is desired to choose the distribution having the largest mean. It is required to find a sample size and a decision rule which will guarantee a correct decision with prescribed probability when the largest mean is sufficiently distinct from the others. R. E. Bechhofer [Ann. Math. Statist. **25** (1954), 16–39; MR **15**, 638] considered the decision rule R which selects the distribution with the largest sample mean. Analogous problems and decision rules have been discussed in other papers. Here it is shown that in a class of problems of this type no decision rule other than R or its analogues can meet the guarantee with a smaller (fixed) sample size.

W. Hoeffding (Chapel Hill, N.C.)

7578:

Greenberg, B. G.; and Sarhan, A. E. Matrix inversion, its interest and application in analysis of data. J. Amer. Statist. Assoc. **54** (1959), 755–766.

Special attention is given to the inversion of matrices which have a structured pattern making possible simplification of the inversion. Special methods are available, when $r=2$ and $r=3$, for a matrix which is diagonal of type r. (A matrix is diagonal of type r if $a_{ij}=0$ for $|i-j| \geq r$ and $a_{ij}=a_{ji}$ for all $i \neq j$.) The authors suggest the use of partitioning in matrices generated from problems involving the fitting of response surfaces. Where the matrix has no special pattern, as is common in regression problems, the authors recommend the modified square root method.

P. S. Dwyer (Ann Arbor, Mich.)

7579:

Pillai, K. C. S.; and Mijares, Tito A. On the moments of the trace of a matrix and approximations to its distribution. Ann. Math. Statist. **30** (1959), 1135–1140.

This paper presents formulas for the first four moments of the sum of s non-null latent roots of a matrix occurring in multivariate analysis. In addition the first four central moments are obtained for the case $s=6$ and formulas for β_1 and β_2 are given. These and the corresponding percentage points are tabulated for special cases and are compared with previous approximate results.

P. S. Dwyer (Ann Arbor, Mich.)

7580:

Walsh, John E. Exact nonparametric tests for randomized blocks. Ann. Math. Statist. **30** (1959), 1034–1040.

"A class of nonparametric procedures for testing the statistical identity of treatments in randomized block experiments is suggested and discussed ... The basic idea is to obtain from each block a statistic that is, under the null hypothesis, symmetrically distributed about zero and then to apply to the set of these statistics a nonparametric test of symmetry about zero." (From the author's summary)

W. Hoeffding (Chapel Hill, N.C.)

7581:

Steel, Robert G. D. A multiple comparison rank sum test: treatments versus control. Biometrics **15** (1959), 560–572.

"A multiple comparison rank sum test, for comparing treatments with a control in a one-way classification with equal numbers of observations, is presented. Both the exact and an approximate distribution are discussed. An example and tables of critical values are given." (Author's summary)

M. Zelen (Washington, D.C.)

7582:

Geisser, Seymour. A method for testing treatment effects in the presence of learning. Biometrics **15** (1959), 389–395.

"np individuals from the population under investigation are assigned at random to p groups, n to a group. The p groups are given p treatments on p different days, in a latin square arrangement. To test the hypothesis that the treatment responses are different, it is inappropriate to use the usual analysis of variance procedures. A method of analysis based on Hotelling's T^2 provides an exact test for the hypothesis of differences among the treatment responses." (From the author's summary)

J. Kiefer (Ithaca, N.Y.)

7583:

Shah, B. V. On balancing in factorial experiments. Ann. Math. Statist. **29** (1958), 766–779.

Consider a $S_1^{m_1}S_2^{m_2}\cdots S_k^{m_k}$ factorial experiment such that (i) any contrast belonging to the interaction involving q_i factors at S_i levels ($i=1, 2, \cdots, k$) is estimated with the same variance (say) $\sigma^2/\theta_{q_1q_2\cdots q_k}$; (ii) the covariance between the estimates belonging to different orders of interactions is zero; and (iii) the block size is constant. The author proves that such a factorial design is also a partially balanced incomplete block design. Treatments having p factors at the same level are pth associates. [These results are related to the paper by C. R. Rao, Sankhyā **17** (1956), 165–174; MR **18**, 683, who developed the two factor case.]

M. Zelen (Washington, D.C.)

7584:

Box, G. E. P.; and Draper, Norman R. A basis for the selection of a response surface design. J. Amer. Statist. Assoc. **54** (1959), 622–654.

This very interesting paper is concerned with a general approach to experimental design, as illustrated by the following simplest case. Let y be a response variable, of form $\eta(x)+\varepsilon$, where ε is an error term, and x a design variable, in one or more dimensions. It is assumed that, within a comparatively large "operability region" O, $\eta(x)$ equals a second degree polynomial. The purpose of the experiment is to represent $\eta(x)$, within a smaller "region of immediate interest" R, by a first degree polynomial $\hat{y}(x)$, to be fitted to the observations on y by the method of least squares. The error $\hat{y}(x)-\eta(x)$ in this representation will consist of two parts: the random component $\hat{y}-E\hat{y}$ and the bias component $E\hat{y}-\eta$. While most work in experimental design has been concentrated on the former (in accordance with the assumption that the true regression of y on x be linear), the present study emphasizes the bias component.—First, the case of a one-dimensional x

is studied. After defining appropriate measures for the two error components, the authors derive classes of optimal designs when (a) variance alone, (b) bias alone, and (c) both are present, and of the same order of magnitude. It turns out that the optimal design in case (c) is nearly the same as in case (b). The possibility of testing the linear hypothesis is also kept in mind. The analysis is repeated, with due modifications, in the case of a k-dimensional x.

C. Elfving (Helsinki)

7585:

Bose, R. C.; and Carter, R. L. Complex representation in the construction of rotatable designs. Ann. Math. Statist. **30** (1959), 771–780.

The necessary and sufficient condition that a two factor arrangement in which the uth sample point is specified by (x_u, y_u), $u=1, 2, \cdots, N$, should be rotatable of order d is $\sum_u z_u^{a-b}\bar{z}_u^b=0$ for all integers a and b satisfying $0<a\leq 2d$, $0\leq b<a/2$, where z_u is the complex variable x_u+iy_u and $\bar{z}_u$ is its complex conjugate. [This is a simpler and entirely equivalent result (for the case of two factors) of G. E. P. Box and J. S. Hunter, Ann. Math. Statist. **28** (1957), 195–241; MR **19**, 75; cf. eq. 35.] The authors use this result to combine circular rotatable arrangements into configurations possessing a higher order of rotatability.

M. Zelen (Washington, D.C.)

7586:

Chernoff, Herman. Sequential design of experiments. Ann. Math. Statist. **30** (1959), 755–770.

The sequential designs considered are based on a class of experiments and, for each E in that class, a sequence $\{x_n(E)\}$ of i.i.d. random variables. Letting $O_n=(E_n, x_n)$, where $x_n=x_n(E_n)$ and E_n is chosen according to a distribution λ_n depending only on $O_{n-1}=(O_1, \cdots, O_{n-1})$, the designs consist of an experimental design $\Lambda=\{\lambda_n\}$ and a sequential stop function N (i.e., a function on sequences $\{O_n\}$ to non-negative integers with $[N\geq n]$ depending only on O_{n-1}).

The paper concerns the asymptotic risk attainable with designs (Λ, N), as a constant incremental sampling cost $c\to 0$, in classifying the family of distributions of $x_1(E)$ into one of two specified classes, $\mathscr{P}_1$ and $\mathscr{P}_2$, with fixed positive loss functions associated with the possible misclassifications.

Let $z_n(E)$ denote the value of $\log dP_E/dQ_E$ at $x_n(E)$ for $(P, Q)\in\mathscr{P}_1\times\mathscr{P}_2\cup\mathscr{P}_2\times\mathscr{P}_1$, $z_n=z_n(E_n)$, $I(P, Q, \lambda)=\int\int z_1(E)dP_E d\lambda$ and, assuming the existence for each O_n of this maximum likehood estimate, let $\hat{P}_n$ attain $\sup_P \inf_Q \sum_1^n z_i=S_n$. Attention centers on the designs $(\hat{\Lambda}, \hat{N})$ where $\hat{\lambda}_{n+1}$ maximizes $\inf_Q I(\hat{P}_n, Q, \lambda)$ and $[\hat{N}>n]=\bigcap_1^n[S_i\leq -\log c]$.

The main development assumes the $\mathscr{P}_i$ and the class of E are finite and, for each E and P, $0<\int z_1^2(E)dP_E<\infty$. Letting $\hat{R}(P)$ denote the risk associated with $(\hat{\Lambda}, \hat{N})$ and the obvious classification based on $O_{\hat{N}}$, while $R(P)$ denotes any risk associated with (Λ, N), the main theorems assert that $\hat{R}(P)\sim c(-\log c)/\sup_\lambda \inf_Q I(P, Q, \lambda)$ and, if $\max_P R(P)/\hat{R}(P)$ is bounded as $c\to 0$, $R(P)\gtrsim\hat{R}(P)$.

J. Hannan (East Lansing, Mich.)

7587:

Derman, C.; Johns, M. V., Jr.; and Lieberman, G. J. Continuous sampling procedures without control. Ann. Math. Statist. **30** (1959), 1175–1191.

Considered are two modifications of step (c) in Dodge's sampling plan (CSP–1) for continuous production [same Ann. **14** (1943), 264–279; MR **5**, 130]. This step concerns action to be taken when a defective partial inspection sample unit is found. The changes are: removal from production flow of uninspected items in the sampled segment containing the defective (CSP–4); (effective) replacement of this segment by one clear of defects (CSP–5). In both cases this is followed by 100 percent inspection starting with the first item following this segment, continuing as in step (a) of CSP–1 until i consecutive non-defective items are observed, then resuming partial inspection. Partial inspection for the modified plans is taken to mean sequential random selection of one unit from each successive segment of k units (random sampling) until a defective is discovered. An average outgoing quality limit (AOQL) for a sampling plan is defined to be the smallest number which is exceeded with probability zero, under each member of a class of production processes, by the lim sup as $m\to\infty$ of the fraction defective in outgoing product after m cycles of the plan. AOQL's are found in terms of i and k and shown to be attainable, first without assuming the process to be in control and then with this assumption. With partial inspection taken to mean that each unit is sampled with probability $1/k$ $(k\geq 1)$ (probability sampling) until a defective is discovered, it is shown that an AOQL (without assuming control) exists for CSP–1 and is the same as that found by Lieberman [same Ann. **24** (1953), 480–484; MR **15**, 240] under the assumption of random sampling.

M. Skibinsky (West Lafayette, Ind.)

7588:

★Dodge, Harold F.; and Romig, Harry G. Sampling inspection tables: Single and double sampling. 2nd ed., revised and expanded. A Wiley Publication in Applied Statistics. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xi+224 pp.

A collection of sampling inspection tables with a description of their use was brought together in the first edition of the book. The second edition contains the tables of the first edition plus a comprehensive collection of probability of acceptance curves called Operating Characteristic curves (OC curves). Some of the additions and changes made in the second edition are: (i) an expansion of the Introduction; (ii) a new chapter on the operating characteristic curve of a sampling plan; (iii) an addition of two sections giving the OC curves for single and double sampling plans for the single and double sampling AOQL (Average Outgoing Quality Limit) tables; and (iv) an addition of a section providing OC curves for a general set of single sampling plans for sample sizes $n=500$ or less and acceptance numbers $c=0$, 1, 2, and 3, based on binomial probabilities. Minor changes in the content of the first three chapters were made. The size and format of the second edition was changed for more effective presentation of the OC curves.

W. T. Federer (Ithaca, N.Y.)

7589:

Nasr, Saad K. On some problems of machine interference. J. Roy. Statist. Soc. Ser. B **21** (1959), 106–113.

"The problem of interference arises when a team of operatives is attending a set of machines, these being assumed to have a certain servicing factor. In this paper

an estimator of the servicing factor and a simple expression for the machine efficiency are introduced. A loss function is also given, with a method for determination of an appropriate number of operatives to be applied for the attendance of the machines." (Author's summary)

H. Teicher (Lafayette, Ind.)

7590:

Sargan, J. D. The estimation of relationships with autocorrelated residuals by the use of instrumental variables. J. Roy. Statist. Soc. Ser. B **21** (1959), 91–105.

The author supposes the coefficients a_i of the relation $\sum_{i=1}^{n} a_i x_{ij} = E_j$ to be functions of $p(<n)$ parameters θ_j. The residuals E include both measurement errors and shock terms, and are non-autocorrelated. If variables u_{jt} $(j=1, 2, \cdots, N)$ are known to be uncorrelated with the E's, then this fact is used to obtain estimates of the θ's. In one formulation of the problem the quadratic form in the a's, $aM_{xu}M_{uu}^{-1}M_{ux}a'$ is minimised, where the M's denote sample covariance matrices. In another, the same form divided by $aM_{xx}a'$ is used.

The asymptotic covariance matrix of the θ estimates is calculated, and the author discusses the questions of multiple solutions, unidentifiability, confidence regions, relation of the method to maximum likelihood, and certain particular cases. *P. Whittle* (Cambridge, England)

NUMERICAL METHODS

See also 7731.

7591:

★Воробьев, Ю. В. Метод моментов в прикладной математике. [Vorob'ev, Yu. V. The method of moments in applied mathematics.] Biblioteka Prikladnogo Analiza i Vyčislitel noĭ Matematiki. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 186 pp. 4.35 rubles.

In the method of Galerkin, if the required function can be expanded by the convergent series $x=\sum_1^\infty a_k\phi_k$, then for purposes of approximation it is replaced by a truncated series. The success of the method depends strongly upon the choice of the ϕ_k. The method of moments limits the choice in the following way: If A is an operator in Hilbert space, and z_0 an element of the space, the sequence $z_i=A^iz_0$ $(i=0, 1, \cdots, n)$ is formed, and A is replaced by the operator A_n in n-space such that $z_i=A_n{}^iz_0$ $(i=0, 1, \cdots, n-1)$, $E_nz_n=A_n{}^nz_0$, where E_n projects into the space of $z_0, \cdots, z_{n-1}$. It is readily shown that $A_n=E_nAE_n$.

General convergence theorems are developed and applications made to a variety of problems, including the solution of integral equations and of differential equations. The exposition is good, and the method seems to be of real value. *A. S. Householder* (Oak Ridge, Tenn.)

7592:

Mahovikov, V. Ĭ. Approximate methods of conformal mapping of two-linked regions. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 257–275. (Ukrainian. Russian and English summaries)

"The proposed methods of conformal mapping of two-linked regions are based on functions derived on employing conformal mapping of corresponding one-linked regions. Each of these methods frequently enables us simply to compose functions of uncomplicated structures which map a definite class of regions." *Author's summary*

7593:

Bandemer, Hans. Berechnung der reellen Eigenwerte einer reellen Matrix mit dem Verfahren von Rutishauser. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 807–814.

The method of Rutishauser [Z. Angew. Math. Phys. **6** (1955), 387–401; MR **17**, 789] is described briefly, then applied to some matrices of orders 3 and 4.

A. S. Householder (Oak Ridge, Tenn.)

7594:

Picone, Mauro. Un metodo di calcolo automatico degli estremi di una funzione o delle soluzioni di un'equazione vettoriale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 373–380.

The author discusses the location of both unconstrained and constrained minima of a function $f(x)=f(x_1, \cdots, x_r)$ of r real variables. If $f(x)$ is twice continuously differentiable, if $f(x)$ has but a single point y at which it has a minimum in the set $D[f(x_0)]=\{x: f(x)<f(x_0)\}$, and if $D[f(x_0)]$ is bounded, he shows quite simply that $\lim x(t)=y$ as $t\to\infty$, where $x(t)$ is the solution of the vector differential equation $dx(t)/dt=-\operatorname{grad} f(x)$, $x(0)=x_0$. A generalization in which $\operatorname{grad} f(x)$ is replaced by a vector function $g(x)$ such that $x\cdot g(x)<0$ whenever $x\neq 0$ is presented. He then treats the location of the minimum of $f(x)$ subject to constraints $\phi_i(x)=0$, $i=1, \cdots, g<r$, by projecting the gradient at $x(t)$ onto the tangent space to the variety $\phi_i(x)=0$, $i=1, \cdots, q$, after starting from a point x_0 on the variety. Finally, as an application he finds roots of vectorial equations by converting such problems to minimization problems. For example, if $F(z)=u(z)+iv(z)$ is analytic, finding a root of $F(z)=0$ is equivalent to minimizing $f(z)=u(z)^2+v(z)^2$. He does not discuss the effects of solving the differential system numerically and of approximating the evaluation of $\operatorname{grad} f(x)$, both frequently occurring difficulties in practice.

J. Douglas, Jr. (Houston, Tex.)

7595:

Glénisson, Y.; et Derwidue, L. Une nouvelle méthode de calcul des zéros des polynômes. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 197–204.

A method is given here for the calculation of all the zeros of a real or complex polynomial. This method is based on the interpolation polynomial of Lagrange and on the expression of the nth power of a zero by a polynomial of degree $n-1$ in this zero. *E. Frank* (Chicago, Ill.)

7596:

Schröder, Johann. Vom Defekt ausgehende Fehlerabschätzungen bei Differentialgleichungen. Arch. Rational Mech. Anal. **3**, 219–228 (1959).

This paper continues recent work by the author [same Arch. **2** (1958/59), 367–392; MR **20** #7392] on which it depends. The interest is in the solution of boundary value problems of the form: $M[u]=f(x, u)$ on $\mathfrak{B}$ and $U_i[u]=\gamma_i$ $(i=1, 2, \cdots)$ on Γ, where $\mathfrak{B}$ is a p-dimensional region with

boundary Γ and M, U_i are linear homogeneous differential operators. The region $\mathfrak{B}$ admits a Green function. In the cited paper, the author considered an iteration process and, for the estimate of the error, he required two functions $u_0(x)$ and $u_1(x)$ related by

$$M[u_1] = f(x, u_0), \quad U_i[u_1] = \gamma_i.$$

If $\mathfrak{B}$ is complicated it is not always easy to obtain these functions, so error estimates are here given in terms of only one approximate solution u_0 and the associated errors

$$-M[u_0]+f(x, u_0), \quad U_i[u_0]-\gamma_i.$$

Some numerical examples are also given.

R. G. Bartle (Urbana, Ill.)

7597:

Blair, A.; Metropolis, N.; von Neumann, J.; Taub, A. H.; and Tsingou, M. A study of a numerical solution to a two-dimensional hydrodynamical problem. Math. Tables Aids Comput. **13** (1959), 145–184.

The authors consider non-steady motion under gravity of two incompressible fluids in a rectangular cell. Initially the fluids are at rest, with fluid of density $\rho=1$ in a trapezoid that forms the upper portion of the cell, and fluid of density $\rho=1/8$ in the complementary lower trapezoid. The equations of motion and continuity are written in Eulerian form and are supplemented by (i) $\rho_t+u\rho_x+v\rho_y=0$. The stream function ψ is introduced and pressure is eliminated to produce a system of two partial differential equations for ψ and ρ. These are solved numerically by converting them to difference equations. An essential feature of the motion is the density discontinuity at the interface between the two fluids, but this was not explicitly taken into account in the choice of difference equations. Accordingly, in the course of calculation the discontinuity gradually becomes spread out, a phenomenon described as "pseudo-diffusion" by von Neumann. One purpose of the work was to see if the motion of the interface could be followed in time from a plot of density contours.

Flow charts and detailed statements of difference equations are given. The principal novelty is the treatment of the density equation (i). In the body of the fluid several alternative forms were used, the choice depending on the signs of u and v. At the boundaries of the cell similar difference versions of $\rho_t+(\rho u)_x+(\rho v)_y=0$ were used, since the difference forms of (i) do not allow the discontinuity to reach one boundary of the cell. A study of the equation $\rho_t+u\rho_x=0$ of one-dimensional flow suggests the stability condition $\Delta t \leq \min(\Delta x/|u|, \Delta y/|v|)$. Computations were carried out on a 15×38 lattice. In graphs of computed results the interface was taken to be the curve of mean $\rho=0.5625$, and the boundaries of the "pseudo-diffusion" band were taken as $\rho=0.5625+0.1750$ (20% original density difference). At the 60th time step (0.339 sec) the band was generally less than two space meshes ($\Delta x=\Delta y=$ 1 cm) wide.

The essence of the numerical work is described in von Neumann's notes on a scheme for accelerating iterative solutions of systems of linear equations (ii) $A\xi=\alpha$ that arise from partial differential equations of elliptic type. A typical iterative method could be taken in the form

$$\text{(iii)} \qquad \begin{pmatrix}\xi^{k+1}\\ \alpha\end{pmatrix} = \begin{pmatrix}G & H\\ O & I\end{pmatrix}\begin{pmatrix}\xi^{k}\\ \alpha\end{pmatrix} \equiv E\begin{pmatrix}\xi^{k}\\ \alpha\end{pmatrix},$$

where G, H, O, and I (unit) are square matrices of the same order as A, and ξ^k is a kth approximation to ξ. (iii) conserves the solution of (ii) if and only if $G=I-HA$, where H is non-singular. To accelerate convergence von Neumann proposed to use as kth approximation a mean $\mathfrak{n}^k=\sum_0^k \alpha_{kl}\xi^l$, or

$$\text{(iv)} \qquad \begin{pmatrix}\mathfrak{n}^k\\ \alpha\end{pmatrix} = P_k(E)\begin{pmatrix}\xi^0\\ \alpha\end{pmatrix}, \quad P_k(E)=\sum_0^k \alpha_{kl}E^l.$$

Plausible arguments are given to suggest that most rapid convergence will be achieved if G is taken to be a Hermitian matrix with characteristic roots between $-1+\varepsilon$ (min) and $1-\varepsilon$ (max), and if $P_k(E)$ (aside from elementary transformations) are taken to be Chebyshev polynomials. It is shown that (iv) converges $(2/\varepsilon)^{0.5}$ times as fast as ordinary iteration (iii). In general von Neumann suggests taking $H=\alpha A^*$, where * denotes conjugate transpose. However, for an appropriate difference scheme for an elliptic partial differential equation for which A is hermitian, take $H=\alpha I$. In either case, the scalar α is used to shift the characteristic roots of G within the desired range.

J. H. Giese (Aberdeen, Md.)

7598:

★Levens, A. S. Nomography. 2nd ed. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. viii + 296 pp. (2 unbound inserts)

The book is a considerably enlarged edition of Levens' well-known *Nomography* [Wiley, New York, 1948]. The number of pages is 296 instead of 176, the number of exercises is 216 instead of 127, the number of figures is 57 instead of 28. Subtitles and increased number of examples contribute to the understanding of the subject presented.

In Chapter 2 is given a very good explanation of the "Projection method" for locating graduations on the scale. Added is Chapter 7 on "Circular nomograms". Chapter 14, corresponding to Chapter 13 of the first edition, is considerably expanded and provided with new examples. Also added is a brief and clear explanation of properties of determinants, enabling the reader to make the necessary manipulations for constructing alignment charts. The new Chapter 15 on "Projective transformations" explains very well the graphical and analytical methods, making the corresponding subject in the classical treatise by d'Ocagne easier to understand. An extremely well written Chapter 16 on "Relationship between concurrency and alignment nomograms with applications to experimental data" contains a very good explanation of rectification of curves.

All in all a book of great value for any one dealing with nomograms.

D. Mazkewitsch (Cincinnati, Ohio)

7599:

Zalts, K. Ya. On the anamorphising multiplier. Latvijas Valsts Univ. Zinātn. Raksti **8** (1956), no. 2, 11–20. (Russian. Latvian summary)

If one considers the system of points $x_i=F_i/H_i$, $y_i=G_i/H_i$, where F_i, G_i, H_i are functions of the variable z_i ($i=1, 2, 3$), the condition that three points are collinear is given by the determinant $(FGH)=0$. In nomographical problems the inverse question has to be solved: to find

for a given functional relation $\Phi(z_1, z_2, z_3)=0$ the necessary and sufficient conditions that it may be represented by a nomogram using aligned points and to find the corresponding functions F, G, H. The necessary and sufficient conditions were given by Gronewall but they turned out to be too complicated for practical work. Sometimes, when the function Φ itself does not satisfy the required conditions, an anamorphising multiplier v exists, so that $v\Phi=0$ satisfies the conditions. The author succeeded in finding the necessary and sufficient conditions for the existence of an anamorphising multiplier for the class of functions

$$\Phi = F(z_1)K(z_2, z_3)+G(z_1)L(z_2, z_3)+H(z_1)M(z_2, z_3).$$

If P_i' indicates differentiation with respect to z_i, the conditions are

$$\begin{vmatrix} K & L & M \\ K_2' & L_2' & M_2' \\ K_2'' & L_2'' & M_2'' \end{vmatrix} = 0, \qquad \begin{vmatrix} K & L & M \\ K_3' & L_3' & M_3' \\ K_3'' & L_3'' & M_3'' \end{vmatrix} = 0,$$

while the anamorphising multiplier is a factor of

$$\begin{vmatrix} K & L & M \\ K_2' & L_2' & M_2' \\ K_3' & L_3' & M_3' \end{vmatrix}.$$

The equations of the third and fourth nomographical rank in three variables are discussed in detail.

E. M. Bruins (Amsterdam)

7600:

Plainevaux, J. E. Intégration graphique des équations différentielles du 1er ordre dans un plan de phases. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 470–484.

Utilisation graphique dans le plan des (y, y') de la méthode des trapèzes. Amélioration possible par doublement du pas. *J. Kuntzmann* (Grenoble)

7601:

Epfelbaum, Gilda; et Laurent, Pierre-Jean. Extension de la table des zéros et des extrémas de la fonction d'Airy, Ai(x). Chiffres **2** (1959), 35–37. (English, German and Russian summaries)

This gives 3-decimal values of the zeros a_s of $x\,\mathrm{Ai}(x)=\mathrm{Ai}''(x)$ with corresponding 5-decimal extreme values of the derivative $\mathrm{Ai}'(a_s)$, and also 3-decimal values of the zeros a_s' of $\mathrm{Ai}'(x)$, with corresponding 6-decimal extreme values of $\mathrm{Ai}(a_s')$. Values are given for $s=51(1)100$. Here $\mathrm{Ai}(x)$ is a solution of $y''=xy$ such that $y\to 0$ as $x\to+\infty$; all zeros have $-37.7>a_s, a_s'>60.1$. (The authors' summary is not well-phrased.) *J. C. P. Miller* (Madison, Wis.)

7602:

Weeg, Gerard P. Numerical integration of

$$\int_0^\infty e^{-x}J_0\left(\frac{\eta x}{\xi}\right)J_1\left(\frac{x}{\xi}\right)x^{-n}dx.$$

Math. Tables Aids Comput. **13** (1959), 312–313.

The integral $I(\eta, \xi, n)$ of the title is given to 5-decimals, with error maxima up to about 15 units quoted, for $n=0$, 1, and $(\xi, \eta)=(1, 1)$, $(1, 3)$, $(1, 6)$, $(3, 1)$, $(3, 3)$, $(6, 1)$, $(6, 6)$. *J. C. P. Miller* (Madison, Wis.)

7603:

Sherry, M. E.; and Fulda, S. Calculation of Gamma functions to high accuracy. Math. Tables Aids Comput. **13** (1959), 314–315.

This gives 35-figure values of $\Gamma(\frac{1}{3})$, $\Gamma(\frac{2}{3})$, $\ln\Gamma(\frac{1}{3})$, $\ln\Gamma(\frac{2}{3})$. The final relative error is claimed as within $\pm 3\times 10^{-15}$. *J. C. P. Miller* (Madison, Wis.)

7604:

Longman, I. M. A short table of $\int_x^\infty J_0(t)t^{-n}dt$ and $\int_x^\infty J_1(t)t^{-n}dt$. Math. Tables Aids Comput. **13** (1959), 306–311.

This gives tables, with an outline of the methods of computation, of the integrals of the title, denoted respectively by L_n and M_n.

Six-decimal values are given (when they exceed $\frac{1}{2}\times 10^{-6}$) for $n=0(1)22$, $x=1(1)10$. Negligible values are reached, for example at or before $x=2$, for $n>$ about 17, before $x=4$, for $n\geqq 9$ or 10, and before $x=10$ for $n\geqq 6$.

J. C. P. Miller (Madison, Wis.)

7605:

Rogers, William M.; Hall, William J.; and Powell, Robert L. Tables of transport integrals: a supplement. J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 23–30.

This paper extends the *Tables of Transport Integrals* by W. M. Rogers and R. L. Powell [Circular 595, Nat. Bur. Standards, Washington, 1958; MR **20** #2853] to the cases $n=18$, 20. The function tabulated is defined as

$$J_n(x) \equiv \int_0^x \frac{e^z z^n dz}{(e^z-1)^2}$$

and is given to 6 figures for $x=.2(.2)50$.

J. C. P. Miller (Madison, Wis.)

7606:

★Numerical Computation Bureau, Tokyo. Tables of Whittaker functions (wave functions in Coulomb field), Part 2. Report No. 11. Numerical Computation Bureau, Tokyo, 1959. 52 pp. $3.00.

This volume is the third one on Whittaker functions to come from the Numerical Computation Bureau in Tokyo [Report No. 8 (1954); No. 9 (1956); MR **16**, 175; **18**, 421]. The entire volume is this time devoted to the imaginary part of $W_{ir,1/2}(-ix)$. More accurately the table gives 5S values of $F=F(r, k)$ where F is the imaginary part of

$$e^{-\pi r}\frac{\Gamma(1-i/k)}{|\Gamma(1-i/k)|}\,W_{ir,1/2}(-2ikr)$$

for $k=0(.01)1$ and $r=0(.1)10$, together with central second differences with respect to k and r. For $k=0$ we have $F(r, 0)=tJ_1(2t)$, $t=(2r)^{1/2}$. The calculations were made for integer values of r via the power series. Subtabulation by means of the differential equation brought the interval down to .1. *D. H. Lehmer* (Berkeley, Calif.)

COMPUTING MACHINES

See also 7812.

7607:

★**Jeenel, Joachim. Programming for digital computers.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1959. viii+517 pp. $12.00.

This book is not a text, but rather a monograph on programming techniques. The scope of the techniques is restricted, except for a few remarks in the seventh chapter to one-address machines and command languages. The result is that, except for scattered references, the techniques are limited to direct machine language coding as of early 1954. Thus, subroutines and their calling sequences, pseudo-codes, their interpreters, and the relationship of the interpretive mode to machine simulation are covered, and in many spots covered rather well; but relative coding is barely covered by the description of a relocater, and symbolic addressing, coding, translation, assembly, and compiling are completely jumbled. There is nothing at all on special language techniques for information processing, or on techniques for translation, or on techniques for automatic storage allocation or on generators and macro-instructions, let alone anything on their theory.

The author justly gives prime importance to flow-charting as a necessary preliminary to coding but does not give due stress to the intermediate stages between flow charting and final coding, the simulation of which introduces symbolic programming, simple translation, and simple allocation of storage, the bases for symbolic assembly programs of the 'SAP' type. Furthermore the author does not warn of some of the limitations in flow-charting, of the need, for example, for process-charts with indications for simultaneous action and involving different processing units; such indications are needed in large data-processing problems and are poorly presented by the strictly sequential flow-charts we are still accustomed to. Furthermore, the author over-emphasizes the stratification of looping levels; in many scientific computations the presentation in the flow-chart of the strata of loops does correspond exactly to the organization of the program and provides a simple means whereby the running time may be computed, if the loops are all run to a fixed number of iterations rather than to a tolerance or to a sentinel. However, many scientific problems, especially those of the simulation of process or of control type, weave in and out of many loops according to some intricate priority pattern, and there are no clear levels. In data processing and automatic coding this lack of stratification is even more marked.

The chapters of the book cover, in order, general information on general purpose computers,' Programs: addressing in memory', 'Programs: sequencing', 'Programs: storage and input-output', 'Problem preparation', 'Languages', 'Operating, testing and checking', and 'Problem planning'. There are nine appendices on 'Number systems', 'Precision and scaling', 'Sorting by pairs', 'Conversion between decimal and binary representation', 'Sorting by merging', 'A procedure for matrix inversion', 'Optimum programming', 'Alternative plans for a network problem', and 'Memory as a buffer'. The appendices are rather sketchy.

There is an attempt to keep the discussion general and machine independent except for mention of the variety of instructions caused by differences in number representation. However, in spite of the mention of various complementing notations for negative numbers, there is no reference to the variety of shift instructions entailed. Nor is there a discussion of the variety of division instructions and their peculiarities. Nor, in the case of floating-point representations, is there an explanation of the excess notation for the exponent, or the pro's and con's involved in packing the exponent in front of the mantissa, or the pro's and con's of normalization after each operation. In passing, it might be remarked that the Newton square root example on page 181, presumably for fixed point operation, will overflow at instruction 102 whenever X is greater than 1/4.

Rough spots appear in the presentation of general principles. For example, the distinction made on pp. 174–175 between transfers on condition and transfers on indication would appear to depend only upon the equipment available; either can always be simulated by programming, so that from the programming point of view there is no such distinction, as the author makes clearly evident in his examples. Similarly, the definitions of 'preset' and 'computed' branches are concerned only with the local time in the immediate neighborhood of a variable exit. From the programming point of view it would be better to show their essential identity by talking about variable exits, or 'variable remote connections', to use the von Neumann-Burks-Goldstine terminology, or even more generally about variable instructions and methods of achieving them. This would include indexing of addresses in a loop, with which the author begins, address modification to simulate a programmed switch, as in the present context, and the most primitive method of all, completely ignored by the author, namely, the substitution of instructions such as jumps from a store of constants into a properly placed empty storage position in the middle of a program. Such an approach would make the use of 'jump tables' mentioned by the author (p. 209) natural and obvious rather than a tricky invention.

It is evident by now that although the book presents an impressive armory of technical tricks, in this reviewer's opinion they could have been organized to appear more natural if the proper basic principles of 'loop control' machines had been presented. As it is, the reviewer feels that, for all its size, the book is too little and too late, especially in view of programming texts already available.

S. Gorn (Philadelphia, Pa.)

7608:

Gorn, Saul. On the mechanical simulation of habit-forming and learning. Information and Control **2** (1959), 226–259.

The study of so-called learning machines may be approached from two points of view. From the first and more speculative of these, one envisions machines capable of performing a variety of "creative", as opposed to routine, tasks such as playing chess, proving mathematical theorems, or writing sonnets. From the second, machines are seen as tools to be used by psychologists for the simulation of mental processes, much as they are used by physicists for simulating a satellite by solving the differential equations that describe its orbit in accordance with some appropriate model.

The author's paper is written from the second point of view. It provides the reader both with a brief introduction

to those properties of digital computing machines that enable them to be used as simulators, and with a survey of models formulated by psychologists. A major portion of the paper is devoted to a mathematical analysis of certain properties of these models. The difficulties presented by such an analysis even in the simplest non-trivial cases demonstrate the great potential usefulness of the simulation technique, although no one who is used to closed form solutions or even to estimates of asymptotic behavior will feel that he can achieve the same degree of understanding of a process through simulation as through formal manipulations of mathematical models for the process of the type presented by the author.

Since most recent publications in the field are based on the first point of view, the author's article is a welcome addition to the literature. Interested readers will find further references in the article, and in earlier articles by Shannon [Proc. I.R.E. **41** (1953), 1234–1241; MR **15**, 902] and by Oettinger [Phil. Mag. (7) **43** (1952), 1243–1263; MR **14**, 587]. *A. G. Oettinger* (Cambridge, Mass.)

7609:

Vandenberg, Steven G. Medical diagnosis by computer: recent attempts and outlook for the future. Behavioral Sci. **5** (1960), 170–174.

7610:

Uhr, Leonard. Intelligence in computers: the psychology of perception in people and in machines. Behavioral Sci. **5** (1960), 177–182.

MECHANICS OF PARTICLES AND SYSTEMS

See also 7275, 7331, 7700, 7776.

7611:

★Полак, Л. С. (Редактор). Вариационные принципы механики. [Polak, L. S. (Editor). Variational principles of mechanics.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 932 pp. 37.25 rubles.

Der umfangreiche Band enthält eine sorgfältig zusammengetragene Auswahl der wichtigsten Arbeiten zu den Variationsprinzipen der Mechanik. Angefangen mit einer Notiz von P. Fermat über Lichtbrechung aus dem Jahre 1662 ist in chronologischer Folge bis zu einer Arbeit von P. A. Dirac aus dem Jahre 1950 das wichtigste aufgenommen, das dem Herausgeber geeignet erschien, den historischen Gang der Entwicklung erkennen zu lassen. Neben den jetzt höchst seltsam und unwissenschaftlich anmutenden metaphysischen Spekulationen von Maupertuis findet man natürlich die Klassiker Bernoulli, Euler, d'Alembert, Lagrange, Hamilton, Jacobi, Helmholtz, Hilbert und viele andere bedeutende Forscher vertreten.

Man kann dem Herausgeber und Kommentator um so dankbarer für diesen Sammelband sein, als er sich neben einer sorgfältigen bibliographischen Zusammenstellung noch die Mühe gemacht hat, eine gründliche, hundert Seiten umfassende Übersicht über die Entwicklung der Variationsprinzipe und ihre Bedeutung in der modernen Physik zu geben. Nicht nur der historisch interessierte Theoretiker wird das Werk begrüßen, man wird es vielmehr auch als Arbeitsunterlage verwenden können, da sich das Zurückgehen auf die Quellen gelegentlich als notwendig erweist. *K. Magnus* (Stuttgart)

7612:

Magnus, K. Der schwere symmetrische Kreisel in kardanischer Lagerung. Ing.-Arch. **28** (1959), 184–198.

Der Einfluß kardanischer Aufhängung auf die Bewegung eines schweren symmetrischen Kreisels wird für den bisher nie streng untersuchten Fall vertikaler äußerer Kardanachsen untersucht. Es gelingt, einige charakteristische Unterschiede zu dem entsprechenden Fall nichtkardanischer Aufhängung herauszuarbeiten, wie gegenläufiger Sinn von Nutations- und Präzessionsbewegung unter geeigneten Anfangsbedingungen, die Bedingungen für die Existenz regulärer Präzessionsbewegungen, sowie die Stabilitätsbereiche für den aufrechten und hängenden Kardankreisel exakt und für andere Stellungen qualitativ mittels der bekannten Methode des Phasenporträts zu ermitteln. Da Reibungseinflüße unberücksichtigt bleiben, hat man ein Energieintegral, und es lassen sich aus den in einem geeigneten Koordinatensystem angegebenen Lagrangeschen Bewegungsgleichungen zwei Impulsintegrale ermitteln. Mittels dieser drei Zwischenintegrale lassen sich die drei Eulerschen Winkel formal durch Quadraturen als Funktionen der Zeit ermitteln. Bezeichnet indes ϑ den Drehwinkel des Innenwinkels um die innere Kardanachse, so ergibt sich mit $u=\cos\vartheta$ die entscheidende Differentialgleichung

$$\left(\frac{du}{dt}\right)^2 = U(u), \tag{1}$$

wo die "Kreiselfunktion U" im Gegensatz zum nichtkardanischen Fall eine rationale Funktion fünften Grades darstellt, die im wesentlichen von fünf Parametern abhängt. Das hat zur Folge, daß weder (1) noch die Differentialgleichung der Umkehrfunktion

$$\frac{dt}{du} = \frac{1}{\sqrt{U(u)}} \tag{2}$$

durch bekannte Funktionen gelöst werden können. Man ist daher auf numerische Auswertung von (1) angewiesen, was der Verfasser auch in einigen charakteristischen Fällen via Auswertung von (2) unternimmt.
K. Matthies (Columbia, S.C.)

7613:

Valcovici, Victor. Propriétés minimales du mouvement des systèmes. Rev. Math. Pures Appl. **3** (1958), 191–206.

L'auteur définit une variation du mouvement réel du mobile A_i sur sa trajectoire Γ_i par deux fonctions arbitraires de t, l'une vectorielle d'espace $\delta\bar{r}_i$, l'autre scalaire δt (en précisant les conditions de régularité auxquelles il les sommet), la variation étant dite synchrone si δt est identiquement nulle. Les $\delta\bar{r}_i$, soumises à la seule condition de compatibilité avec les liaisons à l'instant t:

$$\sum_i \bar{a}_{ij}\delta\bar{r}_i = 0 \quad (i = 1, \cdots, n; j = 1, \cdots, m),$$

ont un degré d'arbitraire nécessaire et suffisant pour déduire de l'équation de d'Alembert $\sum_i (\bar{F}_i - m_i\bar{a}_i)\delta\bar{r}_i = 0$ les équations explicites du mouvement $m_i\bar{a}_i = \bar{F}_i + \sum_j \lambda_j\bar{a}_{ij}$ (auxquelles, comme on sait, on adjoint pour la

détermination des $\bar{r}_i$, λ_j le système définissant les liaisons: $\sum_i \bar{a}_{ij}\bar{r}_i + \alpha_j = 0$). Mais il y a en outre une fonction arbitraire δt dont on peut profiter pour obtenir certaines propriétés minimales du mouvement: il suffira,

$$f(\bar{r}_1, \cdots, \bar{r}_n, \bar{v}_1, \cdots, \bar{v}_n, t)$$

étant une fonction donnée, de prendre pour δt une fonction satisfaisant à l'équation intégro-différentielle

$$\int_{t_0}^{t_1} \left(\sum_i \nabla_{\bar{r}_i} f \, \delta\bar{r}_i + \sum_i \nabla_{\bar{v}_i} f \left(\delta\bar{v}_i - \bar{v}_i \frac{d}{dt}\delta t \right) + \frac{\partial f}{\partial t}\delta t + f\frac{d}{dt}\delta t \right) dt = 0$$

(où $\bar{v}_i = d\bar{r}_i/dt$), et en particulier annulant la quantité sous le signe $\int$ dans tout l'intervalle (t_0, t_1). C'est ainsi que dans le cas des forces de quasipotential V, l'énergie cinétique étant T, l'intégrale $\int_{t_0}^{t_1} (T+U)dt$ est stationnaire même par rapport à des mouvements non synchrones si δt est astreint à la condition

$$\int_{t_0}^{t_1} \left((U-T)\frac{d}{dt}\delta t + \frac{\partial U}{\partial t}\delta t \right) dt = 0$$

et cela même pour des liaisons non holonomes.

Chaque propriété minimale de cette espèce pourrait être interprétée comme un "principe variationnel": par exemple, en choisissant $f = 2T$, on obtient le "principe de moindre action" avec une condition bien moins restrictive pour δt que celle qu'on lui associe d'habitude. On peut ainsi, dit l'auteur, prolonger à l'infini la suite des "principes variationnels", principes valables d'ailleurs aussi bien pour des liaisons non holonomes. *M. Janet* (Paris)

7614:

Bogoiavlenskii, A. A. On particular cases of motion of a heavy rigid body about a fixed point. J. Appl. Math. Mech. **22** (1958), 873–906 (622–645 Prikl. Mat. Meh.).

The equations of motion of a rigid body, under gravity, rotating about a fixed point O are completely soluble by quadrature in certain familiar cases, e.g., when the body has an axis through O of complete kinetic symmetry. In 1888 Kovalevskaia discovered a new case of complete solubility, and in subsequent years a number of other cases were found with explicit particular (but not general) solutions. The present paper reviews and adds a little to this interesting work, entering into a fair amount of detail.

T. M. Cherry (Melbourne)

7615:

Gerasimov, I. S. The asymptotic integration of a differential equation arising in a self-preserving problem in dynamics. Dokl. Akad. Nauk SSSR **126** (1959), 727–729 (Russian); translated as Soviet Physics. Dokl. **4**, 582–584.

7616:

Grindei, Ioan. Sur les systèmes dynamiques correspondants. Acad. R. P. Romîne. Fil. Iaşi Stud. Cerc. Şti. Mat. **9** (1958), no. 2, 189–197. (Romanian. Russian and French summaries)

"Soient (A) et $(\bar{A})$ deux systèmes dynamiques, dont les forces Q_i et $\bar{Q}_i$ sont fonctions des coordonées. Notons par (B) ce que devient (A) à la suite de la transformation (C). ... Dans cette Note, nous avons trouvé les systèmes correspondants (B) d'un système (A) à deux degrés de liberté et avec des forces non nulles.

Pour la correspondance de première espèce (qui conserve les géodésiques) on utilise les conditions établies par T. Levi-Civita [Ann. Mat. Pura Appl. (2) **24** (1896), 255–300] tandis que pour la correspondance de deuxième espèce (qui ne conserve pas les géodésiques) on utilise les conditions établies par J. E. Wright [ibid. (3) **16** (1909), 1–26], conditions retrouvées par nous d'une autre manière, plus simple." (Du résumé de l'auteur)

L. W. Green (Minneapolis, Minn.)

7617:

Pawlikowski, A.; and Szczurówna, W. On the canonical formalism for dynamical systems with constraints. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 759–763.

The authors consider a dynamical system with n degrees of freedom, described by $n+\nu$ generalized coordinates which obey ν equations of holonomic constraints and a Lagrangian which is a quadratic form of the first time derivatives of the unconstrained coordinates. They give explicitly a method of transition to canonical formalism when the constrained coordinates are functions only of the unconstrained coordinates. This method is applied to obtain the Hamiltonian for each of two dynamical systems composed of N identical particles interacting through a two-body potential with different constraints. These are the classical analogies of the extended and the basic collective Hamiltonians of the theory of Bohm-Pines and Migdal-Galitsky. *L. C. Maximon* (Washington, D.C.)

7618:

Beĭlinson, A. A. The application of the method of functional integration to the construction of a fundamental solution of the Focker-Planck-Kolmogorov equation. Dokl. Akad. Nauk SSSR **128** (1959), 876–879. (Russian)

As a basic relationship in the description of a dynamic system by means of the state probability density functions satisfying the Focker-Planck-Kolmogorov equation may be used the Langevin type equation, $F(x, \dot{x}, \cdots, x^{(s)}, \tau) = \phi(\tau)$, where $\phi(\tau)$ is a random process, so-called white noise. The case when F is a linear function has been solved previously. In the present work the solution of the Focker-Planck-Kolmogorov equation is obtained for the case when F is linear in its higher derivative, by means of a functional integral which is close to the integral in the sense of Wiener's measure. The known results for the linear case are contained in this solution.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

7619:

Skowroński, Janisław; and Ziemba, Stefan. The problem of boundedness of motion in certain mechanical systems. Proc. Vibration Problems **1959**, no. 1, 67–81. (Polish and Russian summaries)

Unter Verwendung eines Satzes von Demidovitch wird gezeigt, daß ein nicht-autonomes schwingungsfähiges System mit positiver Dissipationsenergie unter bestimmten, dem zeitabhängigen Erregerglied auferlegten Bedingungen eine beschränkte Bewegung ausführen kann. Diese Grundbewegung ist im Großen beschränkt und

stellt sich bei ganz bestimmten Anfangsbedingungen ein. Jede Nachbarbewegung mit anderen Anfangsbedingungen nähert sich asymptotisch der Grundbewegung. Ist die Erregerkraft periodisch, dann wird auch die Grundbewegung periodisch und hat dieselbe Periode wie die Erregung.
K. Magnus (Stuttgart)

7620:

Andreev, V. D. On a case of small vibrations of a physical pendulum with a moving point of support. J. Appl. Math. Mech. **22** (1958), 1037–1048 (730–737 Prikl. Mat. Meh.).

Bei einem physischen Pendel stimmt die Richtung der Verbindungslinie von Aufhängepunkt und Schwerpunkt bei Erfüllung der bekannten Schuler-Bedingung bei allen Bewegungen des Aufhängepunktes längs der Erdoberfläche mit der Lotrichtung überein, sofern die Anfangsbedingungen geeignet gewählt wurden. Verfasser untersucht Nachbarbewegungen des Pendels um die Lotrichtung und setzt dabei zunächst das Schwerefeld als zentral voraus. Da die Differentialgleichungen für die Nachbarbewegungen periodische Koeffizienten haben, sind instabile Abwanderungen in bestimmten Fällen nicht ausgeschlossen. Hierfür wird eine Abschätzung gegeben. Die Einflüsse eines nicht genau zentralen Schwerefeldes, sowie von Bewegungen des Aufhängepunktes mit vertikalen Komponenten werden abgeschätzt und als klein erkannt. Sollen sie kompensiert werden, dann müssen ganz bestimmte, in der Arbeit berechnete Momente auf das Pendel ausgeübt werden. *K. Magnus* (Stuttgart)

7621:

Godnev, I. N.; and Orlova, I. V. On the relation between the matrix of kinematic coefficients and the inverse kinetic energy matrices in the problem of molecular vibrations. Akad. Nauk SSSR. Optika i Spektrosk. **6** (1959), 583–588 (Russian); translated as Optics and Spectrosc. **6**, 380–383.

"The use of the Lagrange equations with included holonomic constraints to solve the problem of molecular vibrations in coupled coordinates has been analyzed. Formulas have been derived which relate the matrix of the kinematic coefficients A to the T^{-1} and T_0^{-1} matrices, where T and T_0 are the kinetic energy matrices corresponding to the coupled and independent coordinates."
Authors' summary

7622:

Gudzenko, L. I. Small fluctuations in an essentially nonlinear autocorrelation system. Dokl. Akad. Nauk SSSR **125** (1959), 62–65 (Russian); translated as Soviet Physics. Dokl. **4**, 322–326.

Given a determinate one-dimensional dynamic system, the perturbations of a periodic motion are investigated when the particle is additionally subjected to a small stationary stochastic force. Neglecting higher-order terms and making other approximations, the author describes the disturbance in terms of variance and correlation.
S. K. Zaremba (Swansea)

7623:

Caughey, T. K. Hula-Hoop: an example of heteroparametric excitation. Amer. J. Phys. **28** (1960), 104–109.

The motion of a pendulum in a horizontal plane about a point of support which oscillates is studied by the method of averaging. It is stated that the actual mechanism is somewhat more complicated than that described in the paper. No doubt! *J. P. LaSalle* (Baltimore, Md.)

7624:

Breakwell, John V. The optimization of trajectories. J. Soc. Indust. Appl. Math. **7** (1959), 215–247.

The paper presents calculus of variations general theorem containing the usual Lagrange multipliers and applicable to trajectory optimization of such flight parameters as thrust magnitude, thrust direction (for gimballed engines), angle of attack, etc., for a specified "pay-off" requirement such as maximum range, minimum time to altitude, etc. Specific applications are made for optimization of angle of attack with and without thrust direction optimization during boost of a ballistic missile for attainment of maximum range. Re-entry is also treated with and without lift and the effect of the earth's rotation as appropriate. Applications for climbs to maximum speed, minimum time and earliest interception are also made.
M. G. Scherberg (Dayton, Ohio)

STATISTICAL THERMODYNAMICS AND MECHANICS

7625:

Delcroix, J.-L.; et Salmon, J. Quelques propriétés particulières de l'opérateur de collision élastique de Boltzmann. J. Phys. Radium (8) **20** (1959), 594–596.

L'opérateur de Boltzmann

$$J(fF) = \iint (f'F' - fF) g p dp d\varepsilon dV$$

mesurant le taux de variation de $f(\mathbf{v})$ sous l'effet des collisions élastiques avec des particules dont $F(\mathbf{V})$ est la fonction de distribution des vitesses est bilinéaire en f, F. Les auteurs mettent en évidence pour l'opérateur $G\varphi$ déduit de J en supposant F Maxwellienne à la température T et en posant $f = f_0\varphi$,

$$F = \exp\left(-\frac{M}{2kT} V^2\right), \quad f_0 = \exp\left(-\frac{m}{2kT} v^2\right),$$

une propriété variationnelle qui permet de calculer par les méthodes usuelles d'approximation les valeurs propres lorsque le spectre de G est discontinu; ils généralisent ainsi les propriétés connues [Bayet, Delcroix, et Denisse, meme J. (8) **17** (1956), 923–930; MR **18**, 611] dans le cas du gaz de Lorentz. *J. Naze* (Marseille)

7626:

Waldmann, L. Über die Kraft eines inhomogenen Gases auf kleine suspendierte Kugeln. Z. Naturf. **14a** (1959), 589–599.

An aerosol-particle in a gas with a temperature gradient is subject to the action of a force, calculated by Einstein [Z. Phys. **27** (1924), 1–6] under the assumption that the particle is small compared with the mean free path of the gas. This force is expressed in terms of the heat conduction coefficient. When the shape of the particle is spherical the accommodation coefficient does not enter into the picture.

The author generalizes the problem to include the force upon the particle due to the diffusion in an inhomogeneous gas, which force is expressed in terms of the diffusion and accommodation coefficients. The first kind of force, generalized to the conditions in a gas-mixture is again expressed in terms of heat conduction coefficients. In the most general case there appear also the intermolecular forces. *M. Z. v. Krzywoblocki* (Urbana, Ill.)

7627:

Jancel, Raymond; et Kahan, Théo. Mécanique statistique des plasmas: applications des équations de Boltzmann et de Fokker-Planck. Cahiers de Phys. **13** (1959), 289–308.

This paper summarizes extremely briefly the methods of solution of the Boltzmann equation such as the Hilbert-Enskog-Chapman successive approximations, expansions in orthogonal functions, or reduction to the Fokker-Planck equation, with some indications of when each method is applicable. *W. P. Allis* (Cambridge, Mass.)

7628:

Bogolyubov, N. N.; and Tyablikov, S. V. Retarded and advanced Green functions in statistical physics. Dokl. Akad. Nauk SSSR **126** (1959), 53–56 (Russian); translated as Soviet Physics. Dokl. **4**, 589–593.

The authors suggest an approximation scheme in statistical mechanics based on properties of so-called "Green's functions". These are defined as products $\theta(t-t')\langle[A(t), B(t')]\rangle$ where A and B are two arbitrary operators and the $\langle\ \rangle$ symbol means an average over the grand ensemble: $A=Q^{-1}\,\mathrm{Sp}\,[A\exp(-H/\vartheta)]$; $Q=\mathrm{Sp}\,[\exp(-H/\vartheta)]$. It is first shown that these quantities have "spectral representations" similar to those one has for retarded commutators, etc. in field theory [G. Källén, Helv. Phys. Acta **25** (1952), 417–434; MR **14**, 435; H. Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332]. The weight functions appearing in these representations fulfil a system of coupled equations. It is then suggested that this infinite system is converted to a finite system by "some kind of approximation procedure". At this stage the authors say: "The approximation must be chosen with regard to the particular problem under consideration and we do not claim to have a procedure of general applicability." To illustrate their intentions the authors then discuss two particular models, viz., the Heisenberg model of ferromagnetism and the Fröhlich model of superconductivity and show that they can give approximation schemes for these cases which yield the conventional results one has in these models.

G. Källén (Lund)

7629:

Rosenzweig, Walter. Binding effect corrections in the energy-loss distribution functions for charged particles passing through thin absorbers. Phys. Rev. (2) **115** (1959), 1683–1686.

7630:

Pál, L. I.; and Németh, G. A statistical theory of lattice damage in solids irradiated by high-energy particles. Nuovo Cimento (10) **12** (1959), 293–309. (Italian summary)

7631:

Castañs Camargo, M. Quantum statistics as distributions of maximum randomness. An. Real Soc. Españ. Fis. Quim. Ser. A **55** (1959), 305–308. (Spanish. English summary)

"An intuitive meaning of Bose-Einstein and Fermi-Dirac Statistics is obtained by maximising γ (uncertainty or randomness) as applied to the number of particles per cell probabilities in the μ phase space." *Author's summary*

7632:

Kotera, Takeyasu; and Toda, Morikazu. On the theory of Brownian motion and spin relaxation. J. Phys. Soc. Japan **14** (1959), 1475–1490.

"The Brownian motion of a harmonic oscillator is treated quantum-mechanically and the same method is applied to the relaxation of a spin. The method is quite general and applicable to a system displaced from the equilibrium state by an arbitrary amount initially. The oscillator or the spin is interacting with a thermostat composed of a great number of harmonic oscillators. The equation of motion is solved in operator form taking average as to the thermostat to the second order of the interaction parameter. The Langevin equation, the Kramers-Chandrasekhar equation for a harmonic oscillator and the Bloch equation for a static field are derived."

Authors' summary

7633:

Vol'kenšteĭn, M. V. Problems in the theoretical physics of polymers. Uspehi Fiz. Nauk **67** (1959), 131–161 (Russian); translated as Soviet Physics. Uspekhi **2**, 59–81.

The author discusses and reviews the progress made in a number of the most outstanding current problems in the statistical and molecular physics of high polymeric substances. (These substances are chemical molecular structures of high molecular weight which are generated by the linking together by means of chemical valence forces of a limited number of different structural units.) While the choice of the problems discussed is necessarily provisional and the range somewhat limited by the interests of the author and his associates, this review can serve as a valuable guide to workers who wish to enter this field. The specific highly inter-related problems discussed are: (1) the configurational statistics of polymeric chains; (2) physics of the elasticity of rubber; (3) the vitrification of low-molecular-weight liquids and polymers; (4) the crystalline state of polymers.

H. L. Frisch (Murray Hill, N.J.)

7634:

McLennan, James A., Jr. Statistical mechanics of the steady state. Phys. Rev. (2) **115** (1959), 1405–1409.

This paper examines from a general standpoint the way in which the macroscopic theory of irreversible processes can be fitted to the statistical mechanical theory of steady state phenomena. It is shown that the increase in entropy associated with irreversible phenomena can be traced to interactions with the environment, and some formal expressions are derived for the transport coefficients. A quantum-mechanical formulation of the theory is also provided. *H. S. Green* (Adelaide)

ELASTICITY, PLASTICITY

See also 7288, 7360a–b, 7633, 7692.

7635:

Caĭ, I. P. The use of spherical polar coordinates in the theory of elasticity. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 123–133. (Russian)

The author writes down the equations of elastic equilibrium in spherical polar coordinates and then determines solutions of them in the form $u_r=A_1B_1$, $u_\theta=A_2B_2$, $u_\phi=A_3B_3$, where the functions A_1, A_2, A_3 are functions of r alone, and B_1, B_2, B_3 are functions only of θ and ϕ. The functions B_i turn out to be surface spherical harmonics of simple type, and the A_i to be simple functions of r. The process is repeated for the dynamical equations of elasticity on the assumption that each of the components of the displacement vector is proportional to $e^{i\omega t}$, where ω is a constant. In this case the expressions for the A_i are more complicated: when the B_i are surface spherical harmonics of degree n, the A_i can be derived from two functions which are linear combinations of modified Bessel functions of order $n+\frac{1}{2}$. The author does no more than state the expressions for the components of the displacement vector when the equations are separated in spherical polar coordinates—no boundary value problems are considered. *I. N. Sneddon* (Glasgow)

7636:

Tolokonnikov, L. A. Finite plane deformations of an incompressible material. J. Appl. Math. Mech. **23** (1959), 195–210 (146–158 Prikl. Mat. Meh.).

The author presents a complicated version of the simple theory of finite plane strain in incompressible elastic bodies. For a special stress-strain relation, he discusses solutions of stress boundary value problems using complex variable methods. *W. Noll* (Pittsburgh, Pa.)

7637:

Genensky, S. M.; and Rivlin, R. S. Infinitesimal plane strain in a network of elastic cords. Arch. Rational Mech. Anal. **4**, 30–44 (1959).

A general theory is presented for the problem of plane strain in a sheet of fabric consisting of a network formed by two families of parallel, elastic, perfectly flexible cords, which cannot move relative to each other at their points of intersection. The theory is then specialized to the case when the displacement gradients are small compared with unity. The displacements throughout the sheet can be determined when either edge displacements or edge tractions are specified and for problems in which the boundary conditions are of a mixed type.

A. E. Green (Newcastle-upon-Tyne)

7638:

Bufler, H. Zur Theorie der rollenden Reibung. Ing.-Arch. **27** (1959), 137–152.

Für drei verschiedene Fälle werden die Spannungsverteilungen in der Umgebung der Berührungsfläche zweier gegeneinander gepreßter Walzen berechnet. Dabei werden die Zustände des Haftens, Rollens und Gleitens, außerdem gleiche bzw. verschiedene elastische Konstanten der beiden Walzen berücksichtigt. Im Haftfall ergeben sich für die Druck- und Schubspannungsverteilung längs der Berührungsfläche gekoppelte singuläre Integralgleichungen, die von den von Desoyer für denselben Fall abgeleiteten Gleichungen verschieden sind. Auch die tangentialen Randspannungen innerhalb und außerhalb der Berührungszone werden ermittelt. In den Grenzfällen des vollkommenen Haftens und des vollkommenen Gleitens werden geschlossene Lösungen angegeben. Sie werden für zwei Fälle (eine Walze starr bzw. beide Walzen aus gleichem Material) ausgewertet. Dabei zeigt sich, daß bei reinem Gleiten das Druckmaximum gegenüber der Hertz'schen Verteilung verschoben wird, während bei reinem Haften eine Erhöhung des Druckmaximums festzustellen ist. *K. Magnus* (Stuttgart)

7639:

Engl, W. Weiterführung der Sobreroschen Theorie des ebenen Spannungszustandes. Ing.-Arch. **27** (1959), 128–136.

Es werden die Ergebnisse der Arbeit von L. Sobrero über die Behandlung des ebenen Spannungszustandes mit Hilfe hyperkomplexer Funktionen zusammengestellt. Der Zusammenhang zwischen den hyperkomplexen und den komplexen Funktionen wird aufgezeigt und verschiedene Fälle randbelasteter Scheiben behandelt. Die Arbeit nimmt nicht Bezug auf frühere Arbeiten von K. Schmidt [Ing.-Arch. **19** (1951), 324–341; MR **14**, 428] und K. Stahl [ibid. **22** (1954), 1–20; MR **17**, 316], die der Verfasser offensichtlich übersehen hat.

W. Zerna (Hanover)

7640:

Teodorescu, P. P. Über einige räumliche Probleme der Elastizitätstheorie. Apl. Mat. **4** (1959), 225–238. (Czech and Russian summaries)

First the author discusses various methods for solving the equations of linear elasto-statics with the help of stress functions. He then applies one of these methods to derive series solutions of stress boundary value problems for the half space and the thick infinite plate. The prescribed surface tractions are assumed to be normal. Finally, the author indicates how stress boundary value problems for a rectangular parallelopiped can be reduced to solving an infinite system of linear equations.

W. Noll (Pittsburgh, Pa.)

7641:

Teodorescu, P. P. Über die Berechnung des elastischen Halbraumes unter örtlicher Belastung. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 113–121.

Das Problem des elastischen Halbraumes unter örtlicher Belastung ist bereits Gegenstand einer großen Anzahl von Arbeiten. Sie gehen fast alle von der Einzellast als Singularität aus. Die vorliegende Arbeit geht einen anderen Weg, indem von der Tatsache Gebrauch gemacht wird, daß der Spannungs- und Verformungszustand eines räumlichen Problems der Elastizitätstheorie durch 3 biharmonische Funktionen darstellbar ist. Die örtliche Belastung wird in Form eines Fourier Integrals vorgegeben, die 3 Spannungsfunktionen angegeben, sowie die Spannungen und Verformungen berechnet. Die in der Lösung auftretenden Integrale können verhältnismäßig

leicht mit Hilfe von Näherungsmethoden berechnet werden. Das Verfahren wird am Beispiel des elastischen Halbraumes mit Einzellast verdeutlicht.

W. Zerna (Hannover)

7642:

Chakravorti, A. Centre of flexure of a beam of orthotropic material having a section bounded by an ellipse and its major axis. Z. Angew. Math. Mech. **39** (1959), 309–313. (German, French and Russian summaries)

The title problem is solved by a modified Love's method when the load is acting parallel to the straight edge. In the particular case of an isotropic semicircle agreement with the standard results is shown. *S. C. Das* (Madras)

7643:

★Girkmann, Karl. Flächentragwerke: Einführung in die Elastostatik der Scheiben, Platten, Schalen und Faltwerke. 5te verbesserte und vermehrte Aufl. Springer-Verlag, Vienna, 1959. xxxi+632 pp. Ganzleinen: S 432; DM 72; sfr. 73.70; $17.15; £6 2s. 6d.

A treatise on the engineering theory of disks, plates, shells and frameworks. There is a wealth of examples illustrating the various theories.

7644:

Simon, A. W. Theory of beams composed of two elastic materials. Amer. J. Phys. **27** (1959), 500–502.

7645:

Neuber, H. Berechnung des Spannungsverlaufs in Rohrverzweigungen. Z. Angew. Math. Mech. **39** (1959), 213–218. (English, French and Russian summaries)

"A method is presented which for pipebranchings in the form of Y-pipes allows the calculation of the strain which occurs at the endangered points." (From the author's summary) *R. M. Evan-Iwanowski* (Syracuse, N.Y.)

7646:

Chattarji, P. P. On the stresses in twisted composite paraboloids. Bull. Calcutta Math. Soc. **50** (1958), 155–158.

The paper deals with the problem of a composite elastic body of two different isotropic materials formed by two truncated paraboloids placed end to end, and twisted about its axis of revolution by a distribution of shearing stresses on one curved surface, the other curved surface being fixed. *D. C. Gilles* (Glasgow)

7647:

Dutt, S. B. Torsion of a large thick plate with rigidity varying linearly with depth and rigidly fixed at one face. Z. Angew. Math. Mech. **39** (1959), 290–295. (German, French and Russian summaries)

The problems of torsion of a semi-infinite body and a large thick plate previously investigated by Yu [Quart. J. Mech. Appl. Math. **7** (1954), 287–298; MR **16**, 309] are extended to cases in which the rigidity of the plate varies linearly with depth, by using the same method. The interesting result is obtained that, in the case of a semi-infinite body such as the earth, the variation of rigidity has practically no effect on the stresses and displacement at its surface. *Yi-Yuan Yu* (Cambridge, England)

7648:

Shioya, S. The effect of square and triangular notches with fillets on the transverse flexure of semi-infinite plates. Z. Angew. Math. Mech. **39** (1959), 300–308. (German, French and Russian summaries)

"The problem of estimating the effect of a square or triangular notch on the stresses in a semi-infinite thin plate subjected to a bending moment is treated on the basis of Kirchhoff's theory of bending and by using the method of complex representation due to Muskhelishvili. The perturbation method is used to determine the coefficients contained in the bending function of the plate. For some special cases numerical results are given." (Author's summary) *R. M. Morris* (Cardiff)

7649:

Shapiro, G. S. On a rigid-plastic annular plate under impulsive load. J. Appl. Math. Mech. **23** (1959), 234–241 (172–175 Prikl. Mat. Meh.)

A thin annular-shaped plate is clamped along the inner edge and is free along the outer edge. The material of the plate is rigid perfectly-plastic and obeys Tresca's yield condition and associated flow rule. Analysis is given of the dynamic plastic deformation that occurs when the outer edge of the plate is suddenly given a constant velocity for some finite time. Two cases arise and each involves certain distinct phases of motion. In the first phase, a hinge circle travels inwards from the outer edge towards the inner edge. Depending upon the time during which the outer edge is constrained to undergo uniform velocity, this hinge circle may or may not reach the inner edge, and hence two cases arise. Here it is supposed that the former case applies, but the latter one may also be solved with appropriate modifications of the present analysis. In the second phase, there is a stationary hinge circle at the inner edge about which the entire plate bends uniformly. The third phase commences when the outer edge is made free and the plate now comes to rest after further hinge action at the inner edge. The complete solution is given for each phase of the motion.

The author suggests that the present problem could be investigated experimentally. Little attention appears to have been given to annular plate, dynamic plasticity problems. It may be mentioned here that the present reviewer has given the solution of a problem which, in a sense, may be considered 'inverse' to the one treated in the present paper. This problem concerned the impact loading of a thin circular plate made of ductile metal obeying Tresca's yield condition and associated flow rule. The impact loading was produced by a right circular cylinder made of rigid material that impinged centrally on the plate and then moved normal to, and in contact with, the plate with known velocity. [See H. G. Hopkins, *On the impact loading of circular plates made of ductile metal*, D.A.M. Technical Report DA-2598/7, Brown. Univ., Providence, R.I., December 1953.]

H. G. Hopkins (Sevenoaks)

7650:
Prager, William; and Shield, Richard T. Minimum weight design of circular plates under arbitrary loading. Z. Angew. Math. Phys. **10** (1959), 421–426. (German summary)

"The direct design procedures developed by D. C. Drucker and R. T. Shield [see Quart. Appl. Math. **15** (1957), 269–281; MR **19**, 790] have been used recently [see E. T. Onat, W. Schumann and R. T. Shield, same Z. **8** (1957), 485–499; MR **19**, 1215] to obtain the minimum weight design of circular plates under rotationally symmetric pressure distributions. The sandwich plate and the solid or homogeneous plate were considered and designs for both simply supported and built-in edge conditions were derived. As an extension of this work, the minimum weight design is obtained in this paper for a circular sandwich plate loaded by an arbitrary distribution of pressure. The deformation modes used to obtain the designs are the same as those used by E. T. Onat et al. [loc. cit. sup.] for rotationally symmetric pressure distributions." (From the authors' introduction)

H. G. Hopkins (Sevenoaks)

7651:
Libresku, Liviu. Problems in the theory of thin elastic shells, consisting of elastic isotropic layers symmetrically placed with respect to the center of the surface. Rev. Méc. Appl. **4** (1959), no. 2, 335–349. (Russian)

The known method of solution of statical problems concerning homogeneous shells [see for example: V. V. Novožilov, Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1946**, 35–48; MR **8**, 118; and A. L. Gol'denveĭzer, *Theory of thin elastic shells*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 645] has been generalized to the class of shells described in the title. The author uses another manner of calculation of the bending- and tension-stiffnesses. He introduces also certain resultant values of the Poisson ratio. Finally an inhomogeneous shallow shell subjected to a steady field of temperature is considered.

Z. Kączkowski (Warsaw)

7652:
Jullien, Yves. Vibrations transversales d'une plaque nervurée appuyée sur ses bords. Ann. Télécommun. **14** (1959), 161–164.

7653:
Kurtze, Günther. Bending wave propagation in multilayer plates. J. Acoust. Soc. Amer. **31** (1959), 1183–1201.

The problem of flexural wave propagation and attenuation in multilayer plates is treated by the use of electrical analogues. Equivalent electrical circuits are derived whose behaviour can then be studied by standard circuit analysis methods or by analogue electrical measurements. Two types of multilayer system are investigated in detail. The first is a two plate system enclosing a viscous liquid and the second is a plate which is separated from a series of masses by a porous air-filled layer. Experimental measurements on flexural waves propagated in such systems are found to be in good agreement with the theoretical predictions. It is also shown that such systems may have high loss tangents at particular frequencies.

H. Kolsky (London)

7654:
Sharma, S. K. Flow of a visco-elastic liquid near a stagnation point. J. Phys. Soc. Japan **14** (1959), 1421–1425.

A liquid whose properties are specified by a linear relation between stress, stress-rate and strain-rate is assumed to flow along the z-axis of a cylindrical polar system and to impinge normally on the plane $z=0$ causing a radial flow along this boundary with a stagnation point at the origin. Boundary layer equations are formulated and solved numerically on the assumption of a quartic velocity profile throughout the layer. Physical implications of the results are given.

J. E. Adkins (Nottingham)

7655:
Sharma, S. K. Visco-elastic steady flow. Z. Angew. Math. Mech. **39** (1959), 313–322. (German, French and Russian summaries)

Using a constitutive equation of an incompressible visco-elastic material of a non-linear Maxwellian type the author solves the problems of rectilinear shearing flow, Poiseuille flow and flow in a parallel plate viscometer. Results are stated to be in good agreement with experiments.

A. E. Green (Newcastle-upon-Tyne)

7656:
Cox, A. D.; and Morland, L. W. Dynamic plastic deformations of simply-supported square plates. J. Mech. Phys. Solids **7** (1959), 229–241.

An analysis is given, within the framework of thin plate theory, of the problem of a simply-supported, square plate subjected to a uniformly-distributed rectangular pressure. All effects due to elastic strain, work-hardening and strain-rate are neglected, and, as a further simplification, Johansen's yield criterion is adopted as an approximation to Tresca's. Particular results obtained from the analysis concern the maximum displacement and the total time of motion. Exactly as is the case with other dynamic plastic plate (and beam) problems, the motion involves certain distinct phases and both 'medium load' and 'high load' cases arise. Discussion is given also of the limiting problem of impulsive loading, and the extension to problems of regular polygonal plates is indicated, thereby providing a link with previous work by Hopkins and Prager on problems of circular plates.

As the authors note, the present analysis is notable for its (comparative) simplicity, and this is due essentially to the nature of the boundary conditions and the yield criterion. The extension of the analysis to provide the solutions of similar problems for rectangular plates, for other conditions of edge support and for other yield criteria would be extremely valuable. However, to achieve this, it seems clear that considerable modifications of the analysis would be needed, and their exact nature is not known.

H. G. Hopkins (Sevenoaks)

7657:
Severdenko, V. P.; and Kolosu, V. I. On a field of slip-lines. Dokl. Akad. Nauk SSSR **126** (1959), 964–965 (Russian); translated as Soviet Physics. Dokl. **4**, 708–710.

In this brief note a power series expansion is obtained

for the radius of curvature, R, of slip-lines (Hencky-Prandtl net in plane plastic flow) defined by a pair of equal circular arcs. The independent variables used are the customary curvilinear coordinates α, β, in which R satisfies $R_{\alpha\beta}=R$. The authors derive the series by a not very convincing limiting process; it could in fact be obtained immediately by fitting a series $R=\sum a_{mn}\alpha^m\beta^n$ to Cauchy boundary conditions $R=1$ on $\alpha=0$, $R=1+\alpha$ on $\beta=0$. The expansion converges in the entire α, β plane and is used to compute R at nodal points of a net corresponding to 5° intervals in α, β. However, the convergence is comparatively slow and, in the reviewer's opinion, it is doubtful if the series is preferable to other numerical methods.

Cartesian coordinates of the nodal points are obtained by numerical integration along the slip-lines. The results are tabulated for 15° intervals over angular arcs of 135°; they are concordant with published results obtained by other methods. *R. Hill* (Nottingham)

7658:

Odqvist, Folke K. G. Membrane creep of circular plates. Ark. Fys. **16** (1959), 113–118.

Author develops creep theory of circular membranes (neglecting bending stresses) for finite deflections and Norton's power law for creep. Solution is obtained in form of power series, following Hencky's analysis of elastic circular membranes [Z. Math. Phys. **63** (1915), 311–317]. If exponent n of Norton's law goes to infinity, theory includes perfectly plastic membrane as a limiting case.

W. T. Koiter (Delft)

STRUCTURE OF MATTER

See 7633.

FLUID MECHANICS, ACOUSTICS

See also 7351, 7597, 7702, 7703, 7792, 7793.

7659:

★**Yih, Chia-Shun. On the flow of a stratified fluid.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 857–861. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

In this very interesting paper the author investigated the effect of inertia on steady stratified flows of an incompressible and inviscid fluid. The inertial resistance per unit volume to accelerating forces is not constant if the fluid is stratified with varying density. The analysis of this inertia effect is relatively simple if the flow is steady, which is the case under consideration.

In the absence of gravity, the author was able to show that to every steady stratified flow with a continuously varying density, there corresponds an associated flow of a homogeneous, incompressible, inviscid fluid. Thus, if the actual flow originates from a large container with zero velocity, then the associated flow is irrotational and can be obtained by the potential theory. The velocity of the stratified flow is simply that for the irrotational flow divided by the square root of the local density.

In the flow problems where the gravity effect is important, the author considered the two-dimensional case in particular and derived the non-linear partial differential equation for the stream function of the flow. This equation is applied in turn to treat the steady two-dimensional flow of a stratified fluid in a channel toward a horizontal line sink; an exact solution of the problem is obtained for the case of constant density gradient in the vertical direction at infinity. The most interesting result is that, under the shear influence of inertia effect in the presence of varying density in the stratified flow field, the irrotationality does not persist in general, and, as a result found in a certain range of the Froude number of the flow, the actual stratified flow contains a separated stream line which forms the boundary of a corner eddy, even though the flow is everywhere incompressible and inviscid.

T. Yao-tsu Wu (Pasadena, Calif.)

7660:

Yevdjevich, Vujica M. Analytical integration of the differential equation for water storage. J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 43–52.

The equation of continuity is applied to the flow of water in large storage basins. Under certain assumptions a first order nonlinear differential equation is obtained. Special cases are noted in which it may be integrated by quadratures or reduced to Riccati's or Bernoulli's equations. *E. Pinney* (Berkeley, Calif.)

7661:

Legčenko, I. G. Flow past an elliptic cylinder of an incompressible fluid with a pair of attached vortices. Akad. Nauk SSSR. Izv. Sibir. Otd. **1958**, no. 9, 52–59. (Russian)

7662:

Benjamin, T. Brooke. Shearing flow over a wavy boundary. J. Fluid Mech. **6** (1959), 161–205.

This is a careful and thorough analysis of the flows over a simple harmonic wavy boundary which is either (a) rigid (b) a flexible solid or (c) completely mobile, as if it were the interface with a second fluid. The principal aim is to calculate the normal and tangential stress distributions on the surface. Account is taken of viscosity which is important near the critical layer and near the surface. The analysis is made in curvilinear co-ordinates that follow the contours of the wave train. This allows a linearised form of the equations valid if the maximum wave slope is small, even when large velocity gradients exist near the surface. The analysis is carried out largely without reference to particular forms of the velocity profile, though detailed calculations are given for some representative profiles.

The paper provides a great many results of interest to the theory of wave generation by wind and the generation of vibrations in flexible panels, not all of which can be mentioned here. The analysis for the case of a liquid interface is considered in some detail and amplifies in some respect the work of Miles [same J. **3** (1957), 185–204; MR **19**, 1004]. The shear stresses consist of a mean and a

periodic part, and have their maxima generally in advance of the position of maximum surface displacement. The pressure distribution also has a substantial component in phase with the wave slope, whose magnitude is conveniently expressed in terms of a 'sheltering coefficient' $s=s_1+s_2$, the sum of inviscid and viscous contributions respectively. The interplay between s_1 and s_2 is demonstrated. Generally $s_1 \geqslant s_2$ (as Miles [#7663 below] also has shown), but for very slowly moving waves, s_2 may be negative and dominate s_1, implying that energy must be supplied from the waves to the wind if the configuration is to be maintained! Solutions are also given for the flow over a solid boundary and a sheltering coefficient is determined that compares quite favorably with the rather uncertain experimental values. (The reviewer expects the theoretical estimate to be considerably more reliable.) An important and very valuable work.

O. M. Phillips (Balitmore, Md.)

7663:

Miles, John W. On the generation of surface waves by shear flows. II. J. Fluid Mech. **6** (1959), 568–582.

[For part I, see same J. **3** (1957), 185–204; MR **19**, 1004.] This paper extends an earlier analysis on the generation of surface waves by the instability of a parallel shear flow. In this paper, the original analysis is improved by the inclusion of dominant viscous terms in the Orr-Sommerfeld equation and by applying the boundary conditions on the disturbed rather than the equilibrium surface. The formulation is similar to that of T. Brooke Benjamin [#7662 above] and the two papers can profitably be read in conjunction. Miles also presents here the results of a more accurate numerical integration of the equations, providing estimates of the rate of energy supply to the wave components by this mechanism, which becomes decisively important for larger wind duration.

O. M. Phillips (Baltimore, Md.)

7664:

Miles, John W. On the generation of surface waves by shear flows. III. Kelvin-Helmholtz instability. J. Fluid Mech. **6** (1959), 583–598. (1 plate)

A variational formulation is given for a generalised Kelvin-Helmholtz model with parallel shear flows. It is assumed that the aerodynamic pressure on the wave is in phase with the displacement (not the slope); the instability is of a 'static' type. An integral condition for marginal stability is derived and an approximation found for exponential profiles. This type of instability usually occurs in conjunction with the 'dynamic' instability of the previous paper; and for liquids of small viscosity is usually unimportant. The principal application is to the flow of a light inviscid fluid over a viscous liquid, and leads to productions of critical wind speed and wavelength of amplified disturbance in reasonable agreement with experiment.

O. M. Phillips (Baltimore, Md.)

7665:

Küssner, Hans Georg. Beiträge zum Randwertproblem der elliptischen Tragfläche. Z. Flugwiss. **5** (1957), 50–56.

Im Anschluß an frühere Arbeiten des Verfassers [dieselbe Z. **3** (1955), 1–18; **4** (1956), 21–26; J. Aero. Sci. **22** (1955), 227–230; MR **16**, 878] wird unter Benützung orthogonaler krummliniger Koordinaten und Lamescher Funktionen eine neue Integraldarstellung für den Druck an einer elliptischen Tragfläche gegeben, die in einer inkompressiblen reibungsfreien Flüssigkeit schwingt.

K. Maruhn (Giessen)

7666:

★**van Spiegel, Egbert. Boundary value problems in lifting surface theory.** Thesis, Technische Hogeschool Delft, 1959. vi+138 pp.

The author extends Timman's theory for two-dimensional, unsteady, incompressible flow [J. Aero. Sci. **21** (1954), 230–236, 250; MR **15**, 662] to wings of circular planform. He first determines the Green's function for steady flow as an expansion in oblate spheroidal harmonics, a calculation that is considerably more difficult than that of classical potential theory in consequence of the Kutta condition, which requires the cancellation of the usual edge singularity along the trailing edge. He then obtains lift, moment and induced-drag coefficients for linear and quadratic distributions of downwash and compares these with the earlier results of Kinner [Ing. Arch. **8** (1937), 47–80], Schade [Luftfahrtforschung **17** (1940), 387–400; MR **3**, 285] and Küssner [#7665]. His results are in essential agreement with those of Kinner and Schade (which presumably are less accurate in consequence of satisfying the Kutta condition only at discrete points) but not with those of Küssner. The author attributes the latter disagreement to basic errors in Küssner's theory, an opinion with which this reviewer concurs. He then extends his formulation to unsteady flow, calculates the usual (six) low-frequency stability derivatives, and compares his results with those obtained previously by Kriesels and Schade [Luftfahrtforschung **19** (1942), 282–299; MR **4**, 177] and Garner [Aero. Res. Council Rep. Memo. no. 2885 (1952); MR **18**, 356]. Kochin's work on the problem [Akad. Nauk. SSSR. Prikl. Mat. Mech. **4** (1940), 3–32; **6** (1942), 287–316; transl. as NACA Tech. Memo. no. 1324 (1953), 1–50, 51–93; MR **4**, 228; **14**, 508] is not mentioned. The agreement with the results of Kriesels and Schade is good for some derivatives and poor for others, an explanation of which the author cites errors in Schade's theory [loc. cit.]. An appendix gives a closed-form (but less useful) version of the steady-flow Green's function, following Sommerfield's procedure. Summing up, the author appears to have given a definitive treatment of an important, basic, and previously controversial problem in wing theory.

J. W. Miles (Los Angeles, Calif.)

7667:

van Spiegel, E. Boundary value problems in lifting surface theory. Nat. Luchtvaartlab. Amsterdam. Rep. **22** (1959), 138 pp.

This report is identical with the author's thesis reviewed above.

7668:

Hancock, G. J. Notes on thin wing theory at low supersonic speeds. I. The design of lifting surfaces. Aero. Quart. **10** (1959), 247–265.

A numerical method for determining the distribution of incidence required to produce a pre-assigned wing loading

is developed on the basis of known results from linearized theory. Comparisons with the predictions of "exact linearized theory" for two representative planforms indicate that the method should provide a useful design tool. Further developments are promised in a future paper. *J. W. Miles* (Los Angeles, Calif.)

7669:

Carafoli, E.; et Sandulescu, S. Oscillations harmoniques des ailerons de forme générale en régime supersonique. Arch. Mech. Stos. **10** (1958), 771–782. (Polish and Russian summaries)

By expanding the unsteady supersonic small disturbance equation in powers of a reduced frequency, the author obtains expressions for the pressure distribution over an aileron of arbitrary shape performing harmonic oscillations about a straight hinge line. Different expressions apply accordingly as the hinge line is subsonic or supersonic.

D. A. Spence (Ithaca, N.Y.)

7670:

Cynkova, O. È. Motions of a gas in channels of finite length under variable pressure. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 15–24. (Russian)

This work treats the problem of (linear) gas flow in channels of finite length with the compression and removal of the gas through the openings in the walls and with the variable area of the "closing" cross section at the exit of the channel. Two particular solutions are obtained: (1) when the main flow in the channel is divided by a discontinuity jump into supersonic and subsonic parts; and (2) when the main flow in the channel is sonic.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

7671:

Volkov, R. A. On the jet flow of a viscous fluid. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 318–326. (Ukrainian. Russian and English summaries)

"The laminar jet flow of a viscous liquid is discussed in this article on the basis of the classical equations of the hydrodynamics of a real liquid. The methods of operational calculus are employed to solve these equations, the limiting conditions being the velocities and pressures at the point of jet flow formation. The following types of velocities and pressures are found: (1) outflow of viscous liquid from a vertical flat tube; (2) outflow of viscous liquid from a horizontal flat tube; (3) discharge of viscous liquid from an infinite inclined plane. All flows are considered to be in a field of gravity." (Author's summary)

C. D. Calsoyas (Livermore, Calif.)

7672:

Mhitaryan, A. M. Quantitative analysis of a discontinuous solution of the problem concerning determination of characteristics of a train of waves on a rapid flow. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 1, 47–66. (Russian. Armenian summary)

The problem of roll-waves in an inclined open channel was solved by R. F. Dressler [Comm. Pure Appl. Math. **2** (1949), 149–194; MR **11**, 480] by constructing discontinuous solutions on the basis of the non-linear shallow water theory and the Chezy formula for turbulent resistance. Here the same problem is solved in the same way. The only novelty in the paper seems to be the introduction of a dimensionless parameter ξ whose square is proportional to the "roughness coefficient" r^2 divided by the slope of the channel. The condition for the formation of roll-waves becomes $0<\xi<1$. The results of the calculations are portrayed graphically in somewhat greater detail than in Dressler's paper.

D. H. Hyers (Los Angeles, Calif.)

7673:

Gade, Herman G. Notes on the effect of elasticity of bottom sediments to the energy dissipation of surface waves in shallow water. Arch. Math. Naturvid. **55** (1959), 69–80.

In order to consider the effect of elasticity of bottom sediments on wave decay in shallow water, a simplified problem of a shallow horizontal layer of an ideal fluid overlaying a semi-infinite visco-elastic medium is considered. The wave is supposed to have a sinusoidal surface profile, and non-linear effects are neglected.

The author arrives at a set of equations governing the parameters in the solutions, these equations being finally solved numerically. Discussion of the results shows that large values of $\mu/\rho gH$ (μ shear modulus and ρ density of the visco-elastic medium, H the height of the fluid) is most important in the study of the decay.

D. C. Gilles (Glasgow)

7674:

Ladyzhenskaia, O. A. Solution "in the large" of the nonstationary boundary value problem for the Navier-Stokes system with two space variables. Comm. Pure Appl. Math. **12** (1959), 427–433.

Nella prima parte del lavoro viene data la dimostrazione della univoca risolubilità "in grande" per il sistema di Navier-Stokes, nel caso bidimensionale. Questa dimostrazione coincide con quella già data dall'A. [#7675].

Nella seconda parte si dimostrano alcuni teoremi di stabilità. Riportiamo l'enunciato più espressivo. Indichiamo con Ω l'insieme aperto bidimensionale in cui si studia il sistema; alle soluzioni viene imposta la condizione di annullamento sulla frontiera di Ω. Per un vettore $u=u(x_1, x_2)$ definito in Ω, poniamo $|u|=\{\int_\Omega |u|^2dx\}^{1/2}$ e, se u ha derivate prime di quadrato sommabile e si annulla sulla frontiera di Ω, $\|u\|=\{\int_\Omega \sum_k^2 |\partial u/\partial x_k|^2dx\}^{1/2}$. Poniamo $c_\Omega=\sup_u |u|\,\|u\|^{-1}$. Indicato con ν il coefficiente di viscosità, l'A. chiama "numero di Reynolds generalizzato" (n.R.g.) per una soluzione u dell'equazione stazionaria il valore $2c_\Omega\nu^{-1}\|u\|$. Si ha allora: Una soluzione stazionaria u è stabile se ha n.R.g. minore di 1. Qui la stabilità viene intesa nel senso che, per ogni altra soluzione $v(t)$, si ha $\int_0^\infty \|v(t)-u\|^2dt<\infty$.

G. Prodi (Trieste)

7675:

Ladyzhenskaia, O. A. Solution "in the large" to the boundary-value problem for the Navier-Stokes equations in two space variables. Soviet Physics. Dokl. **123** (3) (1958), 1128–1131 (427–429 Dokl. Akad. Nauk SSSR).

Viene qui dimostrata l'esistenza e unicità "in grande" della soluzione del sistema di Navier-Stokes per un campo bidimensionale arbitrario. Il risultato dà finalmente risposta ad un problema posto e studiato in modo assai profondo da J. Leray [J. Math. Pures Appl. **13** (1934),

331–418]. Precisamente si considera, nel campo piano Ω di frontiera S, il sistema:

$$v_t - \nu\Delta v + \sum_{k=1}^{2} v_k v_{x_k} = -\operatorname{grad} p + f(x,t) \quad (x = (x_1, x_2)),$$

$$\operatorname{div} v = 0$$

nelle funzioni incognite $v=(v_1(x,t), v_2(x,t)$ e $p(x,t)$ con le condizioni ai limiti $v|_S=0$, $v|_{t=0}=a(x)$ (div $a=0$). Si dimostra che esiste una ed una sola soluzione "in grande" (cioè definita per ogni $t\geq 0$ e individuata da dati arbitrariamente grandi), purchè siano finiti gli integrali:

$$\int_\Omega a^2 dx, \quad \int_\Omega [v_t(x,0)]^2 dx, \quad \int_0^t\int_\Omega [f^2+(f_t)^2]dxdt.$$

L'impostazione e la tecnica di soluzione sono assai simili a quelle del precedente lavoro (in collaborazione con A. A. Kiselev) in Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 655–680 [MR **20** #6881]. Qui gioca però un ruolo essenziale la disuguaglianza:

$$\iint u^4(x_1,x_2)dx_1dx_2 \leq$$

$$2\iint u^2(x_1,x_2)dx_1dx_2 \iint (u_{x_1}{}^2+u_{x_2}{}^2)dx_1dx_2$$

valida per ogni funzione $u(x_1, x_2)$ nulla fuori di un compatto.

Una diversa soluzione del problema, in una impostazione ulteriormente generalizzata, è stata successivamente data da J. L. Lions e dal recensore [#7676].

G. Prodi (Trieste)

7676:

Lions, Jacques-Louis; et Prodi, Giovanni. Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2. C. R. Acad. Sci. Paris **248** (1959), 3519–3521.

The existence and uniqueness for all time of turbulent (or weak) solutions to the Navier-Stokes equation for regions in two dimensions is established. The uniqueness proof is based on an estimate due to O. A. Ladyženskaya [#7675] which says that in two dimensions $\int_0^T |u(t)|_{L^4}{}^4 dt$ is finite for all finite T, where $|\ |_{L^4}$ denotes the L^4 norm. Then if u and v are two solutions with the same initial values and $w=u-v$, the authors construct a C^∞ function $\phi(t)$ which approximates $w(t)$ for $0\leq t\leq s$ and 0 for $t>s$ in such a way as to yield an equation for $w(t)$ in which $w'(t)$ does not appear. This leads to the inequality $|w(s)|^2 \leq \text{const} \int_0^s |u(t)|_{L^4}{}^4 |w(t)|^2 dt$ for almost every s in a fixed finite interval ($|\ |$ denoting the L^2 norm), showing that $w(s)=0$ a.e. See also O. A. Ladyženskaya [Comm. Pure Appl. Math. **12** (1959), 427–433].

E. Nelson (Princeton, N.J.)

7677:

Bradshaw, P. Approximate solution of the "inverse problem" of boundary layer theory. J. Roy. Aero. Soc. **64** (1960), 225–226.

7678:

Škadov, V. Ya. Integration of the boundary-layer equations. Dokl. Akad. Nauk SSSR **126** (1959), 730–732 (Russian); translated as Soviet Physics. Dokl. **4**, 585–588.

A general scheme is outlined for solving the laminar boundary layer equations by means of a quasi-power series in the distance along the wall, whose coefficients are functions of the distance from the wall. No numerical results are given and the author appears to be unaware of any of the work on this topic which has been done since 1938, notably that of Görtler [J. Math. Mech. **6** (1957), 1–66; MR **18**, 843].

K. Stewartson (Durham)

7679:

Velihov, E. P. Stability of a plane Poiseuille flow of an ideally conducting fluid in a longitudinal magnetic field. Ž. Èksper. Teoret. Fiz. **36** (1959), 1192–1202 (Russian); translated as Soviet Physics. JETP **9**, 848–855.

The author first shows that, for an ideal fluid with zero viscosity and infinite conductivity, a sufficient condition for stability is that $A^2>1$, where A is the Alfvén number and the velocity profile is arbitrary. The case of finite viscosity (but still with infinite conductivity) is then considered for the case of plane Poiseuille flow. The governing equation in this case is of the fourth-order rather than of the sixth-order as in the case of finite conductivity. It is similar to the Orr-Sommerfeld equation and is solved by the usual asymptotic methods. In addition to the usual critical layer, phase resonance now also occurs at two other points where the wave speed coincides with the local Alfvén speed. The results obtained show that an Alfvén number of the order of 0.1 is sufficient to completely stabilize the flow.

W. H. Reid (Providence, R.I.)

7680:

Menkes, J. On the stability of a shear layer. J. Fluid Mech. **6** (1959), 518–522.

The stability of a heterogeneous shear layer with velocity distribution $U(y)=\tanh y$ and density distribution $\rho(y)=\exp(-2Ly)$ is considered neglecting buoyancy effects. With the disturbances of the form $\exp[ik(x-ct)]$, a solution is found for which $c=-L$ so that the waves move in phase with the heavier fluid. The corresponding neutral curve is the circle $L^2+k^2=0$. The following generalization of Rayleigh's inflection-point theorem is proved: a sufficient condition for self-excited disturbances to exist is that $(\rho U')'$ vanish somewhere.

W. H. Reid (Providence, R.I.)

7681:

Rozin, L. A. Approximate method of computation of the nonstationary turbulent boundary layer in an incompressible liquid. J. Appl. Math. Mech. **22** (1958), 1208–1216 (842–847 Prikl. Mat. Meh.).

Following a method given earlier for laminar boundary layers [Rozin, Prikl. Mat. Meh. **21** (1957), 361–367; MR **19**, 1221] the author transforms the equation of mean motion in an unsteady turbulent boundary layer into an equation for $\phi=(\delta^*/U)G(R^*)$ where $G(R^*)$ is the value of $\rho U^2/\tau_w$ appropriate to steady flow on a flat plate, and $R^*=U\delta^*/\nu$. Then writing $f=(\partial U/\partial x+U^{-1}(\partial U/\partial t))\phi$, which vanishes in the case just named, he conjectures that $(\tau_w/\rho U^2)G(R^*)$ and $(\delta^*/\theta)/(\delta^*/\theta)_{f=0}$ are the same functions of f/f_s as in the laminar case. Here f_s is the value of f at a separation point for which experiments by Loitzansky [Prikl. Mat. Meh. **9** (1945), 433–448; MR **7**, 345] suggest

the value -5. Two cases in which ϕ is given by an ordinary differential equation are studied: (i) External velocity distribution $U=U_0(x/l)k(U_0t/l)$, (ii) Small oscillations of a flat plate parallel to the stream. In both cases it is confirmed that the turbulent boundary layer separates later than would a laminar layer.

D. A. Spence (Ithaca, N.Y.)

7682:

Etkin, B. A theory of the response of airplanes to random atmospheric turbulence. J. Aero/Space Sci. **26** (1959), 409–420.

The treatment of aeroplane behaviour in a turbulent atmosphere was first discussed by Liepmann [J. Aero. Sci. **19** (1952), 793–800] in terms of the two-point velocity correlation function $f(r)$; later Ribner [J. Aero. Sci. **23** (1956), 1075–1077, 1118] applied spectral concepts to the problem. The two approaches are of course equivalent. The present author has used the latter to derive transfer functions for the response to gust velocities w_g of the form $\exp 2\pi i(x_1/\lambda_1+x_2/\lambda_2)$, where λ_1 and λ_2 are wavelengths of an element of the turbulent spectrum. w_g is expanded as a Taylor series up to second-order terms in x_1, x_2, the linear terms being modified to improve the average fit. Comparison with an exact solution due to Sears [J. Aero. Sci. **8** (1941), 104–108; MR **3**, 94] shows this to be accurate except at small wavelengths, for which gust intensities in the atmosphere are small. The spectrum is cut off below a wavelength λ equal to twice the appropriate aeroplane dimension l, resulting in errors in r.m.s. downwash of the order of 3 per cent for a typical value of l/L, where $L=\int_0^\infty f(r)dr$ is the longitudinal scale of atmospheric turbulence (which is assumed to be homogeneous and isotropic). *D. A. Spence* (Ithaca, N.Y.)

7683:

Monin, A. S. The theory of locally isotropic turbulence. Dokl. Akad. Nauk SSSR **125** (1959), 515–518 (Russian); translated as Soviet Physics. Dokl. **4**, 271–274.

The theory of locally isotropic turbulence is developed in terms of relative velocities between one particular particle of the fluid and other particles in its neighbourhood. The motion is locally isotropic if the probability distribution functions for the relative velocities and pressure differences are independent of choice of particle, of orthogonal transformation of the co-ordinates and of time shift. An equation of motion for the relative velocity is then obtained, and with this Kolmogoroff's equation relating the mean square and mean cube of the relative velocity is derived. The procedure is thought to be suited to the derivation of other dynamical equations and one is given for the more general triple product,

$$D_{ij,k}(\xi,\eta)=\overline{v_i(\xi)v_j(\xi)v_k(\eta)},$$

where $v_i(\xi)$ is the relative velocity between points with spatial separation ξ. *A. A. Townsend* (Cambridge, Mass.)

7684:

Matschinski, Matthias. Les moyennes tensorielles et la description théorique du phénomène de turbulence. C. R. Acad. Sci. Paris **249** (1959), 2480–2482.

7685:

Kleĭman, Ya. Z. Certain peculiarities in the motion of mixtures. Akust. Ž. **5** (1959), 157–165 (Russian); translated as Soviet Physics. Acoust. **5**, 158–166.

Compression and rarefaction waves are considered to acoustic approximation in the medium with several components. An explanation is given for peculiarities of the wave-motion of mixtures in comparison with analogous motion of media with one component. In particular, there arises the possibility of the occurrence (in well-defined cases) of a set of waves, travelling one behind another, the number of which is equal to the number of components.

From the author's summary

7686:

Ryzhov, O. S.; and Cherniavskii, S. Iu. On the reflection of weak discontinuities from the throat of a Laval nozzle. J. Appl. Math. Mech. **23** (1959), 114–122 (86–92 Prikl. Mat. Meh.).

"Consideration is given to three-dimensional flows of an ideal gas in Laval nozzles, which have discontinuities in the first derivatives of the velocity components on particular characteristic surfaces. The solutions obtained yield examples of transformations from three-dimensional gas flows into plane-parallel and axial-symmetric flows, and also into other flows in space." (Author's summary)

T. M. Cherry (Melbourne)

7687:

Naugol'nykh, K. A. Propagation of spherical sound waves of finite amplitude in a viscous heat-conducting medium. Soviet Physics. Acoust. **5** (5) (1959), 79–84 (80–84 Akust. Ž.).

Let r be Eulerian radius; x, t Lagrangian variables; and $\xi(x,t)=r-x$. From the Navier-Stokes equations, equation of continuity, linearized energy equation, and early terms of a series for $p(\rho,s)$ the author derives for ξ the equation

$$\text{(i)}\qquad \xi_{xx}+(2/x)\xi_x-(2/x^2)\xi-\xi_{tt}=N_1F+N_2G.$$

Here N_1 and N_2 are very small parameters that depend on coefficients of viscosity, frequency ω of oscillation of a harmonically pulsating spherical boundary

$$\xi=\cos(\omega t-kx_0),$$

etc.; and F and G are functions of ξ and some of its derivatives of the first three orders. By analogy with an exact solution of (i) for $N_1=N_2=0$ the author seeks an approximate solution of the form

$$\text{(ii)}\qquad \xi=A_1f(x)\cos\psi+A_2g(x)\sin\psi,$$

where $f=k^2x_0/x+1/x^2$; $g=kx_0/x^3-k/x$; $A_1=A+N_2a_1(x)$; $A_2=A+N_2a_2(x)$; $A=A_0\exp(-N_2k^2x/2)$; and $\psi=\omega t-\omega x/[1+N_1\xi_t\beta(x)]$. The functions $a_1(x)$, $a_2(x)$, and $\beta(x)$ are determined so the first order terms in N_1 and N_2 vanish when (ii) is substituted in (i). The terms in $\sin\psi_0$, $\cos\psi_0$, and $\sin 2\psi_0$ are found for a Fourier series in $\psi_0=\omega t-kx$ for $v=\partial\xi/\partial t$ with coefficients that are functions of x, and qualitative remarks are made about changes of wave shape. *J. H. Giese* (Aberdeen, Md.)

7688:

Powell, Alan. On the aerodynamic noise of a rigid flat plate moving at zero incidence. J. Acoust. Soc. Amer. **31** (1959), 1649–1653.

The aerodynamic noise resulting from the subsonic flow over a flat rigid plate at zero incidence is analyzed by the author based on similarity concepts. The noise is classified to arise from three origins as (1) edge noise, (2) layer noise and (3) wake noise. The fluctuating surface pressure near the edges (side and trailing edges) of the plate gives rise to the edge noise which is of the dipole nature; its acoustic power is found to be proportional to $U^{4.6}$, where U is the mean relative flow velocity. Quadruple radiation takes place from the turbulence in the boundary layer, producing layer noise and also from the turbulent wake, giving wake noise. The noise power of these two noises is shown to be proportional to $U^{7.6}$. Therefore, the edge noise predominates at low enough Mach numbers; at higher speeds the quadruple noise of the layer and wake will assume greater importance. The latter two are suggested (subject to certain conditions) to have a power spectrum with a single peak, bounded on the low frequency part (emanating from the wake) by a curve with a slope like f^2 and on the high frequency part (from the boundary layer) by a curve which falls off like $f^{-7/4}$, where f is the characteristic frequency.

T. Yao-tsu Wu (Pasadena, Calif.)

7689:

Burger, A. P. The half-plane diffraction problem for harmonic time dependence. Proc. Roy. Soc. London. Ser. A **252** (1959), 411–417.

The solution of the two-dimensional problem of the diffraction of "monochromatic" sound waves by a rigid half-plane reduces to the finding of a solution of

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+k^2\varphi = 0$$

which satisfies the radiation and edge conditions and also the boundary conditions

$$\partial\varphi/\partial y = f(x) \qquad (x > 0,\ y = \pm 0),$$
$$\varphi = 0 \qquad (x < 0,\ y = 0),$$

the screen being the half-plane $x>0$, $y=0$, $-\infty<z<\infty$. Here $f(x)$ is a known function determined by the incident waves. For a soft screen, the boundary conditions are

$$\varphi = g(x) \qquad (x > 0,\ y = \pm 0),$$
$$\partial\varphi/\partial y = 0 \qquad (x < 0,\ y = 0).$$

The author shows that the solution of the rigid screen problem is

$$\varphi(x, y) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\{\varphi_y(\xi, \eta)\}_{\eta=+0}\int_r^{\infty} e^{ikz}\frac{dz}{\sqrt{(z^2-r^2)}}\,d\xi,$$

where $r=\sqrt{\{(x-\xi)^2+y^2\}}$; but $\varphi_y(\xi,+0)$ is known only for $\xi>0$. By use of the other boundary condition, this solution is reduced to

$$\varphi(x, y) = -\frac{1}{\pi}\int_0^{\infty} f(\xi)\int_r^{t+r_0} e^{ikz}\frac{dz}{\sqrt{(z^2-r^2)}}\,d\xi,$$

where $r_0=\sqrt{(x^2+y^2)}$.

The solution of the soft-screen problem is similar but more difficult. To avoid the occurrence of divergent integrals, a method of analytical continuation (following the ideas of Marcel Riesz) is used.

E. T. Copson (St. Andrews)

7690:

Ensminger, Dale. Solid cone in longitudinal half-wave resonance. J. Acoust. Soc. Amer. **32** (1960), 194–196.

"Formulas have been derived for determining particle velocity, particle velocity amplification, stresses, length, and mechanical impedance of sections of solid cones—lateral dimensions small compared with the wavelength—in longitudinal half-wave resonance."

From the author's summary

7691:

Brekhovskikh, L. M. Surface waves in acoustics. Soviet Physics. Acoust. **5** (5) (1959), 3–12 (4–13 Akust. Ž.).

The paper is mainly expository, giving a brief presentation of some fundamental problems connected with acoustic surface waves (comblike corrugated plane surface, cylindrical surface). *C. J. Bouwkamp* (Eindhoven)

7692:

Serrin, James. Poiseuille and Couette flow on non-Newtonian fluids. Z. Angew. Math. Mech. **39** (1959), 295–299. (German, French and Russian summaries)

Rivlin's exact solutions of the Poiseuille and Couette flows of a non-Newtonian fluid are discussed and a method is proposed for determining the value of the cross-viscosity coefficient. It is suggested that certain experimental results of Merrington and Weissenberg can be explained by the work of this paper.

A. E. Green (Newcastle-upon-Tyne)

7693:

Kočina, N. N. Shape-preserving, exact solutions of the equations of magnetohydrodynamics. Dokl. Akad. Nauk SSSR **126** (1959), 528–531 (Russian); translated as Soviet Physics. Dokl. **4**, 521–525.

The author obtains similarity solutions of the equations governing the unsteady motion of a perfectly conducting gas in a magnetic field for one and two dimensions. Explicit solutions are presented and the one-dimensional case is examined in detail.

H. P. Greenspan (Cambridge, Mass.)

7694:

Lyubimov, G. A. The effect of an electromagnetic field on the detonation regime. Dokl. Akad. Nauk SSSR **126** (1959), 532–533 (Russian); translated as Soviet Physics. Dokl. **4**, 526–528.

The author considers a plane, normal, stationary detonation wave in a gas whose electrical conductivity is zero in front of the detonation front and infinite behind it; crossed electric and magnetic fields, E and H, parallel to the front are present. The shock conditions, continuity of mass flux, momentum flux, energy flux, and tangential E, are stated, and the detonation adiabatic is given. It is Lüst's relation with an extra term in $H_{1\tau_1}-H_{2\tau_2}$; τ = specific volume. The results are rewritten for use with cylindrical detonations. The author points out that H and E give access to states of the gas inaccessible in their absence. The paper was proof-read carelessly.

G. E. Backus (Cambridge, Mass.)

7695:
Yosinobu, Hirowo; and Kakutani, Tsunehiko. Two-dimensional Stokes flow of an electrically conducting fluid in a uniform magnetic field. J. Phys. Soc. Japan **14** (1959), 1433–1444.

The authors consider the flow of a viscous incompressible, electrically conducting fluid past a cylinder in a uniform applied field.

A solution is obtained for the very special case of zero conductivity and an infinite value of $\mu H_0^2/\rho V_0^2$ such that the Hartmann number is finite. The presence of the magnetic field leads to a basic equation similar to that which results using the Oseen approximation. Therefore, a solution exists although the corresponding Stokes flow of a nonconducting fluid does not.

H. P. Greenspan (Cambridge, Mass.)

7696:
Yen, K. T. Incompressible wedge flows of an electrically conducting viscous fluid in the presence of a magnetic field. J. Aero/Space Sci. **27** (1960), 74–75.

7697:
Lyubarskiĭ, G. Ya.; and Polovin, R. V. The disintegration of unstable shock waves in magnetohydrodynamics. Ž. Èksper. Teoret. Fiz. **36** (1959), 1272–1278 (Russian); translated as Soviet Physics. JETP **9**, 902–906.

Les auteurs étudient en magneto-dynamique des fluides, les ondes de choc planes, stationnaires, instables, dans le cas où le champ magnétique est peu incliné sur la normale au choc. Après une étude qualitative de la désintégration du choc en plusieurs ondes, ils établissent que cette désintégration donne naissance à deux discontinuités lorsqu'on néglige la composante tangentielle du champ magnétique, et à quatre discontinuités lorsqu'on tient compte de cette composante; dans ce dernier cas la distance entre les discontinuités augmente avec le temps.

H. Cabannes (Marseille)

7698:
Lyubimov, G. A. Shock waves with discontinuous gas conductivity situated in an electromagnetic field. Dokl. Akad. Nauk SSSR **126** (1959), 291–294 (Russian); translated as Soviet Physics. Dokl. **4**, 510–513.

Une onde de choc se propage dans un fluide compressible; avant le choc, la conductivité électrique est supposée nulle, si bien que les champs électrique et magnétique sont indépendants; après le choc, la conductivité électrique est supposée infinie, si bien que l'on a la relation classique $cE+vH=0$. L'auteur étudie les variations après le choc de la pression en fonction de la vitesse.

H. Cabannes (Marseille)

7699:
Pikel'ner, S. B. Structure of a magnetohydrodynamic shock wave in a partially ionized gas. Ž. Èksper. Teoret. Fiz. **36** (1959), 1536–1541 (Russian); translated as Soviet Physics. JETP **9**, 1089–1093.

Dans un gas partiellement ionisé, un choc se compose d'une discontinuité et d'une zone de transition. Les équations relatives à la zone de transition sont résolues de façon approchée dans certains cas particuliers. Lorsque l'écoulement dans cette zone peut être considéré comme stationnaire, l'énergie dissipée est indépendante du degrè d'ionisation.

H. Cabannes (Marseille)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 7409, 7627.

7700:
★Richards, Paul I. Manual of mathematical physics. Pergamon Press, New York-London-Paris-Los Angeles, 1959. xi+486 pp. $17.50.

As is very aptly stated in the preface, the goal of this remarkable book is to condense established theoretical physics, its applications and its mathematical equipment into a single reference volume of reasonable size without sacrificing either logical continuity or fundamentals. For the more classical parts of theoretical physics, and also for a number of recent developments, this goal has been successfully reached, be it at the cost of an extreme brevity of style. The chapters of Part I, devoted to physics, are mechanics, thermodynamics, electromagnetism (with elements of circuit and communication theory), special and general relativity, quantum theory (including three pages of field theory), theory of gases, liquids and solids. The presentation of Part II, mathematics, is adapted to the practical needs of the physicist rather than to giving a survey of modern mathematics. This is entirely in line with the general aim of the book. Knowledge of basic mathematics is assumed. To quote an example the first chapter, on algebra, treats roots of polynomials, curve-fitting, finite series, permutations, elementary topology and monographs. We mention a few other subjects in Part II: integral transforms, differential and integral equations, variational problems and linear programming, probability and game theory, tensor analysis and differential geometry, group theory.

L. Van Hove (Utrecht)

7701:
Shimizu, Masao. On the stability of an electron gas at low density. J. Phys. Soc. Japan **15** (1960), 217–219.

"The stability of an electron gas at low density is discussed on the basis of the energy spectrum of a quasi-particle which has been given by the Bohm-Pines theory."

Author's summary

7702:
Dreicer, H. Electron and ion runaway in a fully ionized gas. I. Phys. Rev. (2) **115** (1959), 238–249.

The transport equations are used to describe the relative flow of electrons and ions in a fully ionized gas under the action of an electric field E. The electrons and ions are assumed to have displaced Maxwellian distributions and the total dynamical friction between these two swarms opposes their relative motion. This force reaches a maximum when the relative velocity is half the average random speed of the electrons and eventually decreases inversely as the square of the relative speed. Thus there is no equilibrium in the presence of an electric field larger than the maximum of the total dynamical friction, and this is termed "runaway" although it is more properly termed "rout" as the entire army of electrons defects. It

describes the situation in an electron accelerator rather than in a plasma.

In the model studied here no mechanism is provided by which the electrons can lose energy. They therefore gain random energy as they fall down the electric field and eventually the random speed reaches twice the drift speed and the rout begins. The paper gives the development in time of this process. *W. P. Allis* (Cambridge, Mass.)

7703:

Dreicer, H. Electron and ion runaway in a fully ionized gas. II. Phys. Rev. (2) **117** (1960), 329–342.

The Boltzmann and Fokker-Planck equations are used in this paper to study the runaway of electrons from a fully ionized plasma. This is quite different from the rout studied in part I, above. A surface $k=0$ in velocity space separates a stable region, $k<0$, in which dynamical friction dominates, from an unstable one, $k>0$, in which the electric field accelerates the electrons indefinitely. The electrons reach the surface $k=0$ from the stable side by diffusion-in-velocity and then evaporate. This paper solves the diffusion equation for the stable region with the boundary condition $f=0$ on a spherical surface approximating the surface $k=0$. *W. P. Allis* (Cambridge, Mass.)

7704:

Buneman, O. Dissipation of currents in ionized media. Phys. Rev. (2) **115** (1959), 503–517.

It is shown that when the drift energy of the electrons in a plasma exceed $0.9\,kT$, they will interact with the ions in such a way as to produce growing waves. The dispersion formula for these waves is found and the conditions for their growth discussed in terms of contours in the complex plane. The growth of a local disturbance is also considered and shown to extend over the region separating the electrons and ions which were located at the original disturbance. When the amplitudes of oscillation become large enough for some electrons to overtake others, the linear analysis becomes inapplicable and resort is had to machine computations of individual orbits. This is done for a one-dimensional model and it is shown how randomization takes place when the crests of the potential waves are sufficient to reflect some electrons. Randomization of the electron energy takes place in some 30 plasma periods. The ions take up the momentum but are not appreciably scattered. This process introduces a "collective resistance" which, for the situations considered, is much larger than the "dynamical friction" resulting from Rutherford scattering processes. *W. P. Allis* (Cambridge, Mass.)

7705:

Baranger, Michel; and Mozer, Bernard. Electric field distributions in an ionized gas. I, II. Phys. Rev. (2) **115** (1959), 521–525; **118** (1960), 626–631.

The probability distribution $W(\mathbf{E})$ of the random field $\mathbf{E}$ in a plasma is calculated by applying the cluster expansion method of Mayer and Mayer to the Fourier transform $F(\mathbf{k})$ of W.

The plasma is considered of uniform density n with a pair correlation function given by the theory of Debye and Hückel. This leads to the cluster integrals $h_1(\mathbf{k})$ and $h_2(\mathbf{k})$ which are calculated numerically in terms of two parameters, $x=ke/r_0^2$ and $y=r_0/\lambda$, where r_0 is very nearly the interparticle distance and λ the Debye length. The series of cluster integrals converges for small values of ke/λ^2, that is, for fields large compared to that at a Debye length from a point charge. The Fourier transform is then given by $F(\mathbf{k})=\exp[\sum n^p h_p/p!]$. The functions $H(\beta)=4\pi\beta^2 W(\beta)$ where $\beta=Er_0^2/e$ are tabulated for $y=0$, 0.2, 0.4, 0.6, 0.8 and that for $y=0$ is identical with the Holtzmark distribution. The distributions $H(\beta)$ for $y\neq 0$ differ according to whether the cluster integrals h_1 and h_2 were calculated at a neutral or at a charged point. They also differ according to whether the random field $\mathbf{E}_H$ of electrons, or the random field $\mathbf{E}_L$ of ions shielded by electrons is considered. Paper I gives the calculations for $\mathbf{E}_H$ at a neutral point, paper II gives the other three cases. The main feature of all four cases is a shift of the maximum of the distributions towards lower fields as r_0/λ increases. *W. P. Allis* (Cambridge, Mass.)

7706:

Kovrizhnykh, L. M. Oscillations of a completely ionized plasma in a cylindrical cavity. Ž. Èksper. Teoret. Fiz. **36** (1959), 839–841 (Russian); translated as Soviet Physics. JETP **9**, 592–593.

Within the framework of magnetohydrodynamics, under the assumption of ideal conductivity, a study has been made of oscillations of a completely ionized plasma in a cylindrical cavity located in a magnetic field. It is shown that such a system is stable and that under certain conditions no waves can propagate along the cavity. *W. P. Allis* (Cambridge, Mass.)

7707:

Bogdankevich, L. S. Radiation from a current-carrying ring moving uniformly in a plasma situated in a magnetic field. Ž. Èksper. Teoret. Fiz. **36** (1959), 835–838 (Russian); translated as Soviet. JETP **9**, 589–591.

Energy losses due to Vavilov-Cerenkov radiation are computed for a current-carrying ring moving uniformly in a plasma perpendicular to its plane and parallel to the external magnetic field. *W. P. Allis* (Cambridge, Mass.)

7708:

Tsytovich, V. N. Interaction between a medium and a ring current incident on it. Soviet Physics. JETP **35** (8) (1959), 983–989 (1407–1416 Ž. Èksper. Teoret. Fiz.).

Let a ring current be incident with a constant velocity on a medium occupying a half-space. The plane of the ring is parallel to the boundary of the half-space, and the direction of motion of the ring current is perpendicular to it. The medium has arbitrary magnetic and electric permeabilities, which may depend on the frequency. The author analyses the spectral distribution of the current in the ring during its motion. In the limiting case of low velocity, conditions under which the ring current is reflected by the medium are investigated, and so are the repulsive forces. *C. J. Bouwkamp* (Eindhoven)

7709:

Thomson, George P. Thermonuclear reactions. Amer. J. Phys. **28** (1960), 221–227.

Review article.

7710:
Kvasnica, Josef. Theory of the Čerenkov effect. Pokroky Mat. Fys. Astr. **4** (1959), 302–308. (Czech)
The paper gives the classical representation of the theory of the Cherenkov effect.
R. M. Evan-Iwanowski (Syracuse, N.Y.)

7711:
Wolf, E. Coherence properties of partially polarized electromagnetic radiation. Nuovo Cimento (10) **13** (1959), 1165–1181. (Italian summary)
The author deals with partial polarization from the point of view of coherence theory. Observing that the usual definition of the Stokes parameters of a quasi-monochromatic wave is ambiguous, he analyses an experiment that brings out the observable parameters of such a wave. This leads to a unique coherence matrix and unique Stokes parameters. *C. J. Bouwkamp* (Eindhoven)

7712:
Sacerdoti, Giancarlo. Studio del moto di una particella carica in un campo magnetico di un solenoide toroidale. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 72–83.
Als Ausgangspunkt dient das magnetische Feld eines toroidalen Solenoids. Die Bewegungsgleichungen eines geladenen Partikels in diesem Feld werden aufgestellt und integriert. Die Trajektorien werden numerisch und graphisch betrachtet. *M. J. O. Strutt* (Zürich)

7713:
Sacerdoti, Giancarlo. Studio del moto di una particella carica in un campo magnetico in presenza di un'onda elettromagnetica piana che si propaga nella direzione del campo, nel caso non relativistico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 530–556.
Der Autor beginnt mit der Aufstellung der nichtrelativistischen Bewegungsgleichungen in einem stationären homogenen Magnetfeld unter der Einwirkung einer polarisierten transversalen elektromagnetischen Welle. Hierauf berechnet er die Geschwindigkeit in einer Periode der Welle. Diese Ergebnisse werden diskutiert und auf $2n$ Perioden erweitert, sowie graphisch dargestellt. Insbesondere wird der Einfluss der Anfangsphase der Welle auf die Geschwindigkeit des Partikels betrachtet. Zum Schluss werden numerische Beispiele gegeben.
M. J. O. Strutt (Zürich)

7714:
Ferrari, Italo. Sul campo di un filo percorso da corrente alternata, in prossimità di un mezzo conduttore. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 406–436.
Auf Grund der Maxwellschen Gleichungen werden für das skalare elektrische Potential und für das elektromagnetische Vektorpotential partielle Differentialgleichungen und Randbedingungen aufgestellt. Die Berechnung der elektrischen und der magnetischen Feldstärke aus diesen Potentialen wird angegeben. Für die Potentiale ergeben sich Integralausdrücke, welche auch für die Feldstärkekomponenten auf unendliche Integrale führen. Die Konvergenz dieser Ausdrücke wird untersucht. Hierauf werden Näherungsausdrücke abgeleitet, welche auf Näherungswerte für die Feldstärken führen. Die Näherungsbedingungen werden angegeben.
M. J. O. Strutt (Zürich)

7715:
Gardner, C. S. Adiabatic invariants of periodic classical systems. Phys. Rev. (2) **115** (1959), 791–794.
The author derives 'appropriate adiabatic invariants, constant to an arbitrarily high order in a slowness parameter', with the help of a succession of contact transformations, and makes application to motion of a particle in a slowly varying electromagnetic field. The presentation of the general theory is a summary which leaves untouched one or two points of substance.
T. M. Cherry (Melbourne)

7716:
Kirstein, P. T. Some solutions to the equations of steady space charge flow in magnetic fields. J. Electronics Control (1) **7** (1959), 417–422 (1960).
"The equations for steady, laminar space charge flow are set up. Sets of particular solutions in prescribed magnetic fields are presented—even when the magnetic field has components perpendicular to the cathode. Flows in a large class of magnetic fields from cylindrical and conical cathodes are shown to result, and the existence of similar flows from spiral sheet cathodes are mentioned."
Author's summary

7717:
Lindsay, P. A.; and Parker, F. W. Potential distribution between two plane emitting electrodes. J. Electronics Control (1) **7** (1959), 289–315 (1960).
"This paper gives the derivation of expressions for the space charge and potential distribution between two plane parallel emitting electrodes. Numerical calculations are carried out for the case when an external potential difference ϕ_{ext} is applied between the electrodes, both being at the same temperature T. It is shown that all possible potential distributions can be represented by a single family of curves, the parameter of the family being $A=\exp(-e\phi_{ext}/kT)$. Thus for a given ϕ_{ext} the same general curve of potential distribution applies, whatever the values of the electrode work functions. Changing the work functions merely shifts the end points of the potential distribution along the curve." *Authors' summary*

7718:
Geranin, V. A. The field of a harmonically magnetized tape. Ž. Tehn. Fiz. **29** (1959), 354–357 (Russian); translated as Soviet Physics. Tech. Phys. **4**, 317–320.

7719:
Sacerdoti, Giancarlo. Studio di un contenitore magnetico a forma di otto. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 62–71.
Für das Magnetfeld in der betrachteten Achterhülle werden Näherungsausdrücke aufgestellt. Diese werden diskutiert und numerisch betrachtet.
M. J. O. Strutt (Zürich)

7720:

May, R. M.; and Schafroth, M. R. Susceptibility of superconducting spheres. Proc. Phys. Soc. **74** (1959), 153–160.

An expression for the magnetic susceptibility of a sphere in a homogeneous applied field is calculated. It is assumed only that a particular Fourier component of the magnetization is proportional to the same Fourier component of the magnetic induction; the constant of proportionality is an arbitrary function of the wave number. The result is analyzed for various forms of the dependence of magnetization on magnetic induction which have been proposed for superconducting metals. A comparison is made with the experimental results for the magnetic susceptibility of colloidal suspensions of superconductors.

D. J. Thouless (Birmingham)

7721:

Bremmer, H. Mode expansion in the low-frequency range for propagation through a curved stratified atmosphere. J. Res. Nat. Bur. Standards Sect. D **63D** (1959), 75–85.

The expansion mentioned in the title is particularly useful in ionospheric propagation at low frequencies. The influence of the earth is taken into account by an appropriate impedance boundary condition. Although the coefficients of the mode expansion are derived in the presence of a curved earth, this curvature proves usually negligible. Details of convergence of the various series expansions are not given. *C. J. Bouwkamp* (Eindhoven)

7722:

Holter, Øivin. On the effect of a magnetic field upon extremely low frequency (ELF) wave packets. Astrophys. Norvegica **6**, 131–145 (1960).

"An expression on a form analogous to the Appleton-Hartree formula for the refractive index is evaluated when the plasma has different components. At extremely low frequencies the best approximation to the exact formula is the Q.T.-approximation, and not the Q.L.-approximation as in the Appleton-Hartree theory. When it is possible to calculate the path of the extraordinary wave-packets in the Q.T.-approximation, it results that the path coincides with the magnetic lines of force. Thus, the more the region where this approximation is valid increases (by lowering the frequency of the wave), the more the packets are guided along the lines of force. No such guiding is present in the ordinary mode." *Author's summary*

7723:

Frahn, Wilhelm E. Beugung elektromagnetischer Wellen in Braunbekscher Näherung. I, II. Z. Physik **156** (1959), 78–98, 99–116.

This paper is an extended version of the author's unpublished *Diplomarbeit* [Aachen, 1951]. The approximation method of Braunbek [same Z. **127** (1950), 381–390; 405–415; **138** (1954), 80–88; MR **12**, 223; **16**, 427] for the diffraction of scalar waves by plane obstacles and apertures is extended to the diffraction of electromagnetic waves. The diffraction of a linearly polarized plane wave normally incident on a circular aperture in a perfectly conducting plane screen (or on a complementary disk) is treated in detail, and the results are compared with those obtained from the exact solution of Meixner and Andrejewski [Ann. Physik (6) **7** (1950), 157–168; MR **12**, 145] and the classical Kirchhoff approximations. In particular, the far field of the circular aperture and the transmission coefficient are evaluated for $ka \gg 1$ (k wave number, a radius of obstacle), and it is concluded that, in contrast to Kirchhoff's, Braunbek's approximation leads in all cases to the correct optical limit. The near fields of the circular aperture and disk are also considered in detail. The author's numerical calculations are illustrated by various diagrams. *C. J. Bouwkamp* (Eindhoven)

7724:

Papadopoulos, V. M. The diffraction and refraction of pulses. Proc. Roy. Soc. London. Ser. A **252** (1959), 520–537.

A method for solving problems involving diffraction of a plane pulse by a perfectly conducting or absorbent infinite wedge is described, including the case of total reflexion. The method is extended so as to be applicable to a perfectly reflecting or absorbent half-plane lying on the surface between two distinct isotropic non-dissipative media. *C. J. Bouwkamp* (Eindhoven)

7725:

Duinker, S. Generalization to non-linear networks of a theorem due to Heaviside. Philips Res. Rep. **14** (1959), 421–426. (French and German summaries)

A classical theorem on linear electromagnetic systems stated by O. Heaviside and proven by H. A. Lorentz is generalized for non-linear reactances. The extension involves electric and magnetic co-energies, in addition to the respective energies, as introduced by E. C. Cherry [Philos. Mag. (7) **42** (1951), 1161–1177; MR **13**, 409]; the two concepts are dual in the sense of interchange of thermodynamically extensive and intensive quantities. The theorem applies only to hysteresis-free reactances {restriction not stated, but implicit} and involves only the sum of energies and co-energies, which represent the products of the respective thermodynamically conjugate quantities. The theorem states that for coherently applied step function sources, at $t=0$, the asymptotic difference between the electric and magnetic products equals that between the total work done by the sources and the "virtual" asymptotic dissipative energy, considered of constant rate from $t=0$. Linearity of dissipance is shown to be essential to the validity of the theorem.

H. G. Baerwald (Albuquerque, N.M.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 7700.

7726:

★Kuntzmann, J. Exemple de problèmes conduisant à un système aux dérivées partielles. Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 64–68. Libreria Eredi Virgilio Veschi, Rome. 1959. xii + 158 pp.

The mathematical formulation of a problem in heat conduction is given followed by a brief discussion of a proposed method for solving the resulting system of four partial differential equations.

D. M. Young, Jr. (Austin, Tex.)

7727:

Ling, Frederick F. A quasi-iterative method for computing interface temperature distributions. Z. Angew. Math. Phys. **10** (1959), 461–474. (German summary)

The author uses a combined analytical numerical method, with several simplifying assumptions, to solve the title problem for frictional heating effects. It is not clear that the problem is close to physical reality or that the numerical method used is the most efficient one; nevertheless, the analytical manipulations and iterative methods are interesting. *C. E. Pearson* (Cambridge, Mass.)

7728:

Tirskii, G. A. Two exact solutions of Stefan's nonlinear problem. Dokl. Akad. Nauk SSSR **125** (1959), 293–296 (Russian); translated as Soviet Physics. Dokl. **4**, 288–292.

One-dimensional heat flow problems are written for the semi-infinite solid $y \geq 0$ when the coefficients in the heat equation are functions of the temperature and when melting occurs at a plane $y = \eta(t)$. The author examines special cases that can be solved by assuming that temperatures depend only on the variable $y/\sqrt{t}$, or on the variable $y - at$, where a is a constant.

R. V. Churchill (Ann Arbor, Mich.)

7729:

Chambré, Paul L. Nonlinear heat transfer problem. J. Appl. Phys. **30** (1959), 1683–1688.

Let u_0 and β denote constants, and f a prescribed function that is not necessarily linear. A method is presented for solving this boundary value problem for the temperatures $u(x, t)$ in a semi-infinite solid: $u_t = u_{xx}$ $(x > 0,\ t > 0)$, $u(x, 0) = u_0$, $u_x(0, t) = \beta f[u(0, t)]$. The surface temperature $u(t) = u(0, t)$ can be determined by solving the non-linear Volterra integral equation

$$u(t) = u_0 - \beta\pi^{-1/2} \int_0^t f[u(\tau)](t-\tau)^{-1/2} d\tau.$$

The temperatures $u(x, t)$ are then given by an integral that contains $u(t)$ and known functions. The author contributes a method for solving the integral equation for $u(t)$. He uses the method of successive approximations on that equation, in conjunction with the Gauss quadrature formula for evaluating the integrals that arise, so as to produce a reliable approximation to $u(t)$ with few steps. Numerical examples include the important case in which $f[u(t)] = [u(t)]^4 - 1$, $u_0 = 0$.

R. V. Churchill (Ann Arbor, Mich.)

7730:

Grigorian, S. S. On heating and melting of a solid body owing to friction. J. Appl. Math. Mech. **22** (1958), 815–825 (577–585 Prikl. Mat. Meh.).

L'auteur étudie deux problèmes de transfert posés par la fusion d'un solide, due au frottement.

L'un de ces problèmes est le suivant: un solide en mouvement dans un fluide visqueux atteint sa température de fusion, du fait du frottement.

L'étude du phénomène nécessite la détermination des vitesses et des températures dans le fluide entourant le solide et dans la couche superficielle en fusion. Le problème est traité dans le cas particulier d'un solide infini, limité par un plan, en contact avec un écoulement uniforme. Les formules résolutives sont données, d'une manière explicite, en particulier l'expression de l'évolution de l'épaisseur de la couche fondue en fonction du temps.

L'autre problème est celui où le chauffage et la fusion du solide sont dus au frottement sur un obstacle solide, la pression de contact étant une fonction du temps donnée. Il y a là une généralisation du problème de Stefan, du fait de la présence de la couche liquide entrainée dans le mouvement et dont l'écoulement doit être déterminé. Le problème est traité—en particulier l'apparition de la fusion est discutée—dans le cas d'une force de pression de la forme $P = C^{te} \cdot t^{-1/2}$, t étant le temps.

R. Gerber (Grenoble)

7731:

Dewey, C. Forbes, Jr.; Schlesinger, Stewart I.; and Sashkin, Lawrence. Temperature profiles in a finite solid with moving boundary. J. Aero/Space Sci. **27** (1960), 59–64.

The authors present a numerical scheme for solving the one-dimensional heat equation in cylindrical coordinates (the radial flow). The main difference between the authors' and the conventional case is the variable thermal conductivity coefficient, which is a function of the temperature. Thus the heat equation becomes non-linear. Although one of the boundaries is variable, the authors make it fixed through a suitable transformation of the independent space variable. Using time-centered differences, they replace the differential equation by difference equations and linearize them assuming that the coefficients of the independent variable are known functions of space and time. The implicit scheme insures stability, although the authors derive the stability criterion as well. The selection of step size aims only at insuring of control the truncation error.

In the second, non-mathematical part of the paper the authors analyse the application of their solution to the re-entry problem. *T. Leser* (Aberdeen, Md.)

QUANTUM MECHANICS

See also 7700, 7709.

7732:

McIntosh, Harold V. On accidental degeneracy in classical and quantum mechanics. Amer. J. Phys. **27** (1959), 620–625.

The close connection existing between group theory and differential equations has been recognized since the work of Sophus Lie. In the physical literature it has been used to advantage primarily for the obvious symmetry groups associated with permutations and with the geometrical transformations of euclidean space. The study of special examples shows the existence of important symmetry groups associated with contact and higher-order transformations in both classical and quantum mechanics. The

author gives a general review of this subject, with references to the literature. Special attention is paid to the connections between the unitary unimodular and rotation groups and their relation to the multi-dimensional harmonic oscillator. Interesting comments are made on the geometrical interpretations of some of the mappings which arise in this theory. *E. L. Hill* (Minneapolis, Minn.)

7733:

Myers, Vernon W. Some exact solutions of the time-dependent Schroedinger equation. Amer. J. Phys. **28** (1960), 114–116.

7734:

Myers, Vernon W. Quantum mechanical treatment of systems with a damping force proportional to the velocity. Amer. J. Phys. **27** (1959), 507–508.

The non-relativistic quantum mechanical analogue of a classical system whose Hamiltonian is given by $H=e^{-\alpha t}p^2/2m+Ve^{\alpha t}$ is described (α being a constant). Attention is drawn to the similarity between the classical and quantum mechanical results for the cases $V=0$ and $V=mgx$. *H. Rund* (Durban)

7735:

Sokolik, G. A. A new class of representations of the full Lorentz group. Ž. Èksper. Teoret. Fiz. **36** (1959), 1098–1102 (Russian); translated as Soviet Physics. JETP **9**, 781–783.

"All representations of the full Lorentz group are found. It is shown that these representations reduce to direct products of spinors belonging to three classes. An attempt is made to interpret isotopic spin in terms of the generalized parity operator without introducing new degrees of freedom.

It is shown that the connection between spin and statistics may not hold for spinors which transform according to commuting representations." (Author's abstract) *A. C. Hurley* (Melbourne)

7736:

★**Mott, N. F. Elements of wave mechanics.** Cambridge University Press, New York, 1958. ix+156 pp. Paperbound: $2.95.

This book has been written to replace the author's *An outline of wave mechanics* [Cambridge University Press, New York, 1930], now out of print. It is intended for students in the final year of an honours course of experimental physics at British universities, and also as an introduction to more advanced textbooks for those who wish to specialise in the subject. The book builds up the elementary theory from experimental facts and omits the more complicated mathematical developments. Examples are included.

7737:

Leruste, Philippe. Quantification du mouvement du rotateur de Nakano. C. R. Acad. Sci. Paris **249** (1959), 2296–2297.

"La quantification du mouvement du rotateur de Nakano conduit à l'équation de Feynman et Gell-Mann." *Résumé de l'auteur*

7738:

Cziffra, Peter; MacGregor, Malcolm H.; Moravcsik, Michael J.; and Stapp, Henry P. Modified analysis of nucleon-nucleon scattering. I. Theory and p-p scattering at 310 Mev. Phys. Rev. (2) **114** (1959), 880–886.

A modified method of analyzing nucleon-nucleon scattering is discussed and applied to proton-proton scattering experiments at 310 Mev. The modified scheme is based on an explicit inclusion in all higher angular-momentum states of the terms contributed by the one-pion exchange process.

Since the only parameter in the Born approximation is the pion-nucleon coupling constant, the modified scheme can also provide a determination of this coupling constant. The application of the modified scheme to p-p scattering at 310 Mev. indicates that the first two of the five best solutions of the conventional phase-shift analysis are more satisfactory than the others. *P.-O. Löwdin* (Uppsala)

7739:

Jin, Y. S. Dispersion relations for associated production. Nuovo Cimento (10) **12** (1959), 455–468. (Italian summary)

The author gives a formal derivation (without considering the analytic continuation of the causal scattering amplitudes into the unphysical region which is necessary for a rigorous proof) of relativistic dispersion relations for the associated production of hyperons and K mesons in pion-nucleon collisions. The usual charge independent strong interactions and isotopic spin assignments for the baryons and mesons are assumed. The derivation starts from the standard expression for the S-matrix element in terms of the matrix element of a retarded commutator [Lehmann, Symanzik and Zimmerman, Nuovo Cimento (10) **6** (1957), 319–333; MR **19**, 1133]. The dispersion relations obtained are written down in both the Breit system and in covariant form, and for both Σ and Λ hyperon production. The main new features of these dispersion relations are the kinematic and isotopic spin complications of the inelastic associated production process. *O. W. Greenberg* (Cambridge, Mass.)

7740:

Vladimirov, V. S. Determination of a region of analyticity. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 275–294. (Russian)

The author improves a theorem by N. N. Bogolyubov and V. S. Vladimirov on analytic continuation of generalized functions [same Izv. **22** (1958), 15–48; MR **20** #6031]. Let F_{rr}, F_{rs}, F_{sr}, F_{ss} be generalized functions each depending on four 4-vectors $x_j=(x_{j0},\cdots,x_{j3})$, $j=1,\cdots,4$; x_{j0} being the time coordinate, $c=1$, and each function invariant with respect to the Lorentz group. It is assumed that $F_{rr}=0$ when $x_1\lesssim x_3$ or $x_2\lesssim x_4$, $F_{rs}=0$ when $x_1\lesssim x_3$ or $x_2\gtrsim x_4$, $F_{sr}=0$ when $x_1\gtrsim x_3$ or $x_2\lesssim x_4$, $F_{ss}=0$ when $x_1\gtrsim x_3$ or $x_2\gtrsim x_4$, the inequality signs meaning earlier than, resp. later than, for every choice of the coordinate system. It is further assumed that the Fourier transforms $\delta(p_1,\cdots,p_4)\tilde{F}_{lm}(p_1,\cdots,p_4)$ of F_{lm} satisfy the conditions $\tilde{F}_{rr}=\tilde{F}_{sr}$, $\tilde{F}_{rs}=\tilde{F}_{ss}$ when $p_1^2<(M+1)^2$, $p_3^2<\gamma^2$; $\tilde{F}_{rr}=\tilde{F}_{rs}$, $\tilde{F}_{sr}=\tilde{F}_{ss}$ when $p_2^2<(M+1)^2$, $p_4^2<\gamma^2$; $\tilde{F}_{lm}=0$ when $(p_1+p_3)^2<(M+1)^2$ or $p_{10}+p_{30}<0$, where $M+1\geq\gamma>1$.

Let V and τ_0 be real numbers satisfying $V<\tau_0\leq 1$, $\tau_0\geq -1-2M$. Under these conditions N. N. Bogolyubov and V. S. Vladimirov proved the existence of a representation $\hat{F}_{lm}(p_1, \cdots, p_4)=\Phi(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, \frac{1}{2}((p_1+p_3)^2)^{1/2})$ where $\Phi(z_1, z_2, z_3, z_4, z_5, t)$ is holomorphic with respect to $z_1, \cdots, z_5$ when $|z_1-M^2|<\rho$, $|z_2-M^2|<\rho$, $|z_3-\tau|<\rho$, $|z_4-\tau|<\rho$, $|z_5+4\Delta^2|<\rho/t^2$, where $V\leq\tau\leq\tau_0$, $0\leq\Delta^2<\Delta_m^2$. The function Φ vanishes when $t<\frac{1}{2}(M+1)$. In the present paper the result is proved for $\Delta_m^2=(1-(1-\tau_0)(2M+2)^{-1})-\tau_0$, and also when Δ_m is the smallest number for which a solution (x, u, v) with $x=\Delta_m$ satisfies a certain system of complicated algebraic equations. In the case $M=7$ the triple $(\tau_0, \gamma, \Delta_m^2)$ may e.g. assume the numerical values (1, 3, 2.56), (0, 2, 2.42), (1, 2, 1.29). *H. Tornehave* (Copenhagen)

7741:

Okubo, S. Note on the conserved current in the weak interactions. Nuovo Cimento (10) **13** (1959), 292–302. (Italian summary)

The Ward identity [J. C. Ward, Phys. Rev. (2) **78** (1950), 182; MR **11**, 632] in electrodynamics is a consequence of the conservation of the electric charge; the general relation between conserved currents and lack of renormalization is studied in this paper in connection with the vector and axial vector currents involved in weak interactions. For a conserved current $j^\mu(x)=\bar{N}(x)\gamma^\mu QN(x)+F^\mu(x)$, where $N(x)$ is the nucleon field operator, Q is a numerical matrix in the spin and isotopic spin indices and $F^\mu(x)$ consists entirely of meson variables, it is shown that $\langle\beta|\int j^0(x)d^3x|\alpha\rangle=u_\beta^*Qu_\alpha$, where $|\alpha\rangle$, $|\beta\rangle$ are the physical one-nucleon states of the same momentum and u_α, u_β are the corresponding spinor amplitudes, provided the meson-nucleon coupling does not involve any derivatives. It is also shown that such conserved axial vector currents and strangeness-violating currents cannot be constructed for non-derivative meson-nucleon couplings and the known mass-spectrum. Several approximately conserved currents are discussed; and credit is given to S. Weinberg for an independent derivation of the main result (on the lack of renormalization) using a different method. *E. C. G. Sudarshan* (Rochester, N.Y.)

7742:

Shekhter, V. M. On the weak-interaction types possible in the scheme of Feynman and Gell-Mann. Soviet Physics. JETP **36** (9) (1959), 403–405 (581–584 Ž. Éksper. Teoret. Fiz.).

This investigation on the possibility of the conservation of currents in "the theory of Feynman and Gell-Mann" is aimed at showing that, under suitable assumptions, the strangeness-conserving vector current alone can be conserved. The assumptions include: (1) currents bilinear in the baryon fields and the meson fields (the author seems to dislike contributions linear in the bosonfields); (2) Yukawa type strong interaction Lagrangians with finite unrenormalized coupling constants and masses; (3) conservation of the current operators rather than for the one-particle matrix elements. A deduction of the results for strangeness-violating currents under much more general conditions is given by S. Okubo [Nuovo Cimento **13** (1959), 292–302]. The author extrapolates the zero-frequency limit theorem (following from the conservation of the vector current) to the large momentum transfer $\Sigma\to\Lambda+e+\nu$ decay to conclude that the vector matrix element vanishes. The confusion between conservation of the current operator and of the current matrix element leads to the "prediction" that the result of the experiment proposed by Weinberg et al. [Phys. Rev. Lett. **1** (1958), 25–27] on the pion-neutrino correlation in $K_{\mu 3}$ decay must be negative. *E. C. G. Sudarshan* (Rochester, N.Y.)

7743:

Dallaporta, N.; and Toyoda, T. On the Dirac equation for baryons. Nuovo Cimento (10) **12** (1959), 593–601. (Italian summary)

The authors discuss an eight component spinor equation similar to the Dirac equation for the electrons which reduces to two uncoupled four component Dirac equations. The invariance properties of the eight component equation under various transformations called boson conjugation and spinor conjugation are described. Interactions with electromagnetic field and with a vector field describing the baryons are introduced in such a way that the corresponding current vectors are conserved.

A. H. Taub (Urbana, Ill.)

7744:

Ceolin, C.; and Toyoda, T. Gauge invariance of baryon equation. Nuovo Cimento (10) **13** (1959), 605–613. (Italian summary)

A non-local but microcausal interaction between a pseudo-scalar field and an eight-component spinor field is determined. This interaction is obtained by expressing the baryon vector field of the eight component spinor equations discussed in the paper reviewed above in terms of a pseudo-scalar π-meson field. *A. H. Taub* (Urbana, Ill.)

7745:

Volkov, D. V. On the quantization of half-integer spin fields. Ž. Éksper. Teoret. Fiz. **36** (1959), 1560–1566 (Russian); translated as Soviet Physics. JETP **9**, 1107–1111.

Let $\psi_a^{1,2}(x)$ be two Hermitean fields which obey the equal time commutation rules,

$$\{\psi_a^1(x), \psi_b^1(x')\} = \delta_{ab}\delta(x-x'),$$
$$\{\psi_a^2(x), \psi_b^2(x')\} = \delta_{ab}\delta(x-x'),$$
$$[\psi_a^1(x), \psi_b^2(x')] = 0;$$

then the field $\psi_a(x)=2^{-1/2}(\psi_a^1(x)+\psi_a^2(x))$ will satisfy the commutation rules,

$$(1)\quad \psi_a(x)\psi_b(x')\psi_c(x'')+\psi_c(x'')\psi_b(x')\psi_a(x)=\psi_a(x)\delta_{bc}\delta(x'-x'')+\psi_c(x'')\delta_{ab}\delta(x-x')$$

of the "Kemmer" type. The field $\psi_a(x)$ will then admit 0, 1 or 2 particles in the same state. Thus, in mom-space, $a_k^*=2^{-1/2}(a_k^{1*}+a_k^{2*})$ so $a_k^*a_k^*|0\rangle=a_k^{1*}a_k^{2*}|0\rangle$, but $a_k^*a_k^*a_k^*|0\rangle=0$ since $(a_k^{1*})^2=(a_k^{2*})^2=0$. In configuration space the wave functions will have mixed symmetries. It is clear that all of the usual invariance properties will also hold for the field $\psi(x)$ as long as $\psi^1(x)$, $\psi^2(x)$ have exactly identical properties. Further only the field $\psi(x)$ in fact need be written and hence provides an example of a relativistic half-integral spin field which does not obey

Fermi-Dirac statistics, which emphasizes that the establishment of the connection between spin and statistics is of a negative sort (half integral spin fields cannot satisfy Bose statistics but need not satisfy Fermi statistics). This paper provides a derivation of the commutation relation (1) from the Schwinger action principle but, remarkably, does not point out how such a field can be represented in terms of the usual type of Fermi-field as outlined above. There is immediate generalization to arbitrarily complicated sorts of statistics for both integral and half integral spin fields.

K. Johnson (Cambridge, Mass.)

7746:

Roy, T. C. Bound states in quantum electrodynamics. Z. Physik **158** (1960), 142–144.

7747:

Fogarassy, B. Interaction between electrons and coherent fields. Acta Phys. Acad. Sci. Hungar. **10** (1959), 305–325. (Russian summary)

7748:

Gatto, R. Problems in the theory of weak interactions. Fortschr. Physik **7** (1959), 147–181.

An excellent critical discussion of the present status of the theory of weak interactions. The topics treated include the nature of the weak interaction currents, isotopic spin transformation properties, hypotheses of universality and of lack of renormalization of the vector strangeness-conserving current and the application of dispersion-theoretic techniques for calculating weak interaction matrix elements.

E. C. G. Sudarshan (Rochester, N.Y.)

7749:

Dallaporta, N. On parity conservation in strong interactions. Nuovo Cimento (10) **13** (1959), 159–172. (Italian summary)

The interaction Lagrangian density for baryons and mesons coupled via strong Yukawa interactions is studied here on the basis of the principle that the interaction terms should possess γ_5 invariance in some sense similar to the γ_5 invariance in weak four-fermion interactions. By adding two such sets of terms with different γ_5 invariance properties, a parity conserving strong interaction Lagrangian density is constructed, which does not possess any "undesirable" symmetries. The coupling constants can be chosen so as to make the interaction charge-independent and to yield mass splittings of the baryons. It is stated that parity conservation may be viewed as a consequence of "the number and disposition of the baryon states in the elementary particle scheme". Incidentally, the title of the paper is misleading.

E. C. G. Sudarshan (Rochester, N.Y.)

7750:

Koide, S. Über die Berechnung von Franck-Condon-Integralen. Z. Naturf. **15a** (1960), 123–128. (English summary)

The Hamiltonian of the normal vibrations of a molecule or crystal is expressed in terms of creation and annihilation operators. Changes in the equilibrium nuclear separations and vibrational frequencies can be represented by transformations which are also expressed in terms of these operators. The procedure offers an elegant method of evaluating Franck-Condon integrals.

A. Dalgarno (Belfast)

7751:

De Laet, C.; et Demeur, M. Remarque sur la théorie de l'excitation coulombienne atomique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 546–552. (English summary)

"The semi-classical treatment of the electromagnetic excitation of a bound electron by a heavy charged particle is discussed. The peculiarities of the impact parameter method are compared with those of the Rutherford orbits method." (Authors' summary) *D. F. Mayers* (Oxford)

7752:

Parker, Paul M.; and Brown, L. Carlton. Energy moment method in quantum mechanics. Amer. J. Phys. **27** (1959), 509–514.

In two previous publications by the authors [Phys. Rev. **100** (1955), 1764–1767; J. Chem. Phys. **27** (1957), 1108–1110] an approach different from the conventional ones to certain non-diagonal quantum mechanical eigenvalue problems had been used. As this method had been described only very briefly, the present paper is devoted to a more detailed and complete description of this process, the so-called energy moment method, for which a discrete energy spectrum is assumed. The procedure relates the physical parameters associated with the description of a quantum mechanical system to energy eigenvalues of the wave equation of the system in such a manner that for relatively simple systems the physical parameters can be evaluated from experimental data without effecting a detailed solution for the roots of the secular equation. This method depends to a very large extent on the algebraic properties of the coefficients of the expanded form of the secular equation. The paper concludes with applications to: (1) nuclear quadrupole interactions in single crystals; (2) the rigid asymmetric rotator.

H. Rund (Durban)

7753:

★Handbuch der Physik (herausgegeben von S. Flügge), Bd. 41/1. Kernreaktionen II: Theorie. Springer-Verlag, Berlin - Göttingen - Heidelberg, 1959. vii + 580 pp. DM 145.00.

This volume contains the four articles reviewed below.

7754:

★Breit, Gregory. Theory of resonance reactions and allied topics. Handbuch der Physik, Bd. 41/1, pp. 1–407. Springer, Berlin-Göttingen-Heidelberg, 1959.

This review article occupies about 400 out of the 560 text pages of the 41st volume of *Handbuch der Physik, Nuclear Reactions II*. After a historical survey, the article starts with an elementary discussion of resonance scattering in a central field. Various treatments of this fundamental phenomenon is given and the relation between the different mathematical methods is exhibited. The

discussion then goes on to more and more complicated cases where the reaction products consist of several fragments. The concept of "channels" in nuclear reactions is introduced and illustrated with simple examples before it is applied to more complicated problems. A very short treatment of γ-ray emission is also included. A very large part of the article is devoted to a discussion of the systematic approach to the theory of nuclear reactions developed by Wigner and others and characterized by the so-called R-matrix. As is well known, the main idea in this approach is to consider the nucleus as a "black box" and to replace the detailed treatment of the nuclear interior by a discussion of the logarithmic derivative of the wave function at the nuclear surface. The author emphasizes among other things that the parameters appearing in this treatment and usually described as resonance energies and reduced width are not always simply and uniquely related to the observed data. A final chapter is devoted to a discussion of more phenomenological theories like the optical model, the theory of stripping reactions, alpha decay theory, etc.

G. Källén (Lund)

7755:

★**Hull, McAllister Hobart, Jr.; and Breit, Gregory. Coulomb wave functions.** Handbuch der Physik, Bd. 41/1, pp. 408–465. Springer, Berlin-Göttingen-Heidelberg, 1959.

The properties of the regular and irregular solutions to the non-relativistic Schrödinger equation with a Coulomb potential are reviewed in a clear and comprehensive way. Most of the discussion is devoted to the case of a repulsive Coulomb field as this is the most important case for applications to nuclear reactions. The authors list various expansion formulae convenient for small and large values of the radius and for small and large values of the energy. A survey of various integral representations is also given. A special paragraph is devoted to approximations like the WKB-method and similar techniques. The article closes with a discussion of methods suitable for numerical calculations and with a list of available tables.

G. Källén (Lund)

7756:

★**Breit, Gregory; and McIntosh, John Stanton. Polarization of nucleons scattered by nuclei.** Handbuch der Physik, Bd. 41/1, pp. 466–495. Springer, Berlin-Göttingen-Heidelberg, 1959.

The authors state in the introduction that they want to give a discussion of the scattering between polarized particles as a tool to obtain information about nuclear structure and, in particular, about the spin-orbit-interaction at high energies. After a short description of the von Neuman statistical matrix the paper deals with essentially two problems, viz. the scattering of spin 1/2 particles by spinless targets and the scattering in a complex potential with one central part and one spin-orbit term. The first problem is developed in some mathematical detail and the discussion includes a treatment of double and triple scattering experiments. The latter problem is handled with the Born approximation technique first developed by Fermi [Nuovo Cimento (9) **11** (1954), 407–411]. The discussion includes also a comparison with experimental data.

G. Källén (Lund)

7757:

★**Breit, Gregory; and Gluckstern, Robert L. Coulomb excitation.** Handbuch der Physik, Bd. 41/1, pp. 496–560. Springer, Berlin-Göttingen-Heidelberg, 1959.

By Coulomb excitation is meant a nuclear reaction where the bombarding energy is so low that no nuclear reaction in the ordinary sense of the word can take place but the target nucleus is excited with the aid of only the electromagnetic forces between the projectile and the target. This type of reaction has proved particularly useful in recent years to obtain information about low lying nuclear levels. The authors' treatment of this phenomenon emphasizes very heavily the semiclassical approach where the projectile is supposed to move along its classical hyperbolic orbit and only the excitation of the target nucleus is described with the aid of quantum mechanics. A detailed discussion of the solution of the complete quantum mechanical problem with the aid of the WKB technique and the Bohr correspondence principle is supplied to justify this treatment. First order quantum mechanical effects are also worked out and shown to be unimportant for practical applications.

G. Källén (Lund)

7758:

Demeur, M.; et Gribaumont, A. Réactions nucléaires de transfert. I, II. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 472–482; **45** (1959), 533–545. (English summary)

In order to describe a nuclear reaction in which a nucleon is transferred from one nucleus to another, the authors consider the transfer of a particle from one one-dimensional potential well to another, the uniform relative motion of the potential wells being described classically.

N. Rosen (Haifa)

7759:

Davidson, J. P. Isotopic spin. Amer. J. Phys. **27** (1959), 457–465.

An elementary exposition of the isotropic formalism and its application in two limited problems is presented. The development emphasizes the similarity of the analogous dichotomic variable problem of spin $\frac{1}{2}$ particles. There are several oversimplifications in the paper and the applications are somewhat elementary.

N. S. Wall (Cambridge, Mass.)

7760:

Becker, R. L.; and Rose, M. E. Polarization of conversion electrons following beta decay. Nuovo Cimento (10) **13** (1959), 1182–1225. (Italian summary)

"The present work contains an essentially complete discussion of the polarization of internal conversion electrons from the K-shell. ... Expressions for the polarization components are given in terms of extensively tabulated 'particle' parameters for the three types of polarization. Numerical results for the longitudinal and transverse in-the-plane polarization parameters are included for mixed as well as for pure multipole transitions. Particular decay schemes involving allowed and first forbidden unique β-transitions are considered in detail and numerical values for these polarizations are given." (From the author's summary)

E. C. G. Sudarshan (Rochester, N.Y.)

7761:
Blinder, S. M. Generalized perturbation method employing nonorthogonal functions. J. Chem. Phys. **32** (1960), 111–115.

7762:
Anderson, P. W. New approach to the theory of superexchange interactions. Phys. Rev. (2) **115** (1959), 2–13.

"The theory of indirect exchange in poor conductors is examined from a new viewpoint in which the d (or f) shell electrons are placed in wave functions assumed to be exact solutions of the problem of a single d-electron in the presence of the full diamagnetic lattice. Inclusion of d-electron interactions leads to three spin-dependent effects which, in the usual order of their sizes, we call: superexchange per se, which is always antiferromagnetic; direct exchange, always ferromagnetic; and an indirect polarization effect analogous to nuclear indirect exchange. Superexchange itself is shown to be closely related to the poor conductivity, in agreement with experiment. By means of crystal field theory the parameters determining superexchange can be estimated, and in favorable cases (NiO, $LaFeO_3$) the exchange integrals can be evaluated with accuracy of several tens of percent. Qualitative understanding of the whole picture of exchange in iron group oxides and fluorides follows from these ideas." (Author's abstract) *A. C. Hurley* (Melbourne)

7763:
Klein, O. On the systematics of elementary particles. Ark. Fys. **16** (1959), 191–196.

Use of both integral and half-integral isotopic spin for both fermions and bosons is felt to be unexpected. Here a generalisation of Kemmer's isotopic-spin scheme is proposed. Strangeness is included by adding a spin $\frac{1}{2}$ fermion field of zero charge and zero isotopic spin thus giving a three-element column matrix in isotopic spin space for the baryon field, and similarly describing the meson field by a 3×3 matrix. The Σ and Ξ particles would have to be regarded as some kind of compound systems. There is similarity between this scheme and one used to describe leptons. *C. Strachan* (Aberdeen)

7764:
Misra, S. P. Fourth order meson equation and neutron-proton scattering. Indian J. Phys. **34** (1960), 92–97.

"We have obtained here the neutron-proton differential scattering cross-section with a fourth order meson equation proposed by Bhabha and Thirring, which was useful in explaining the anomalous magnetic moments of nucleons. We note that for moderate energies the results here disagree as violently with experiments as for conventional meson theory satisfying the Klein-Gordon equation." *Author's summary*

7765:
Károlyházy, F.; and Marx, G. Strong interactions in the four-dimensional isotopic space. Acta Phys. Acad. Sci. Hungar. **10** (1959), 421–428. (Russian summary)

This suggests a form of "global symmetry" in which, in the absence of the K-meson interactions, the other particles transform as representations of an "isotopic" group of 4-dimensional rotations. For this purpose, Λ and Σ hyperons form a degenerate quartet, and cascade particles form part of a sextet with four hypothetical brethren.

The advantage of this scheme is that it allows the K-meson interactions, in a well defined manner, to destroy the 4-dimensional rotation group invariance, leaving invariance under a subgroup of 3-dimensional rotations. *J. C. Taylor* (Rochester, N.Y.)

7766:
Aizu, Ko. Maximal weak interactions. Progr. Theoret. Phys. **22** (1959), 192–206.

This paper systematically enumerates "all possible" weak processes in connection with the "hypothesis of maximal set of interactions, that there is always the weak interaction among any combination of all kinds of particles" (baryon conservation and charge conservation being imposed). According to the author (approximating all transition matrix elements by unity), "it is found that this maximal interaction model is equivalent to the minimal interaction model in the sense that the experimental facts are described equally well by these two models, provided that a selection rule which forbids the appearance of μe pair is added to the former model." But the "model" is so qualitative that any comparison with experiment seems irrelevant. *E. C. G. Sudarshan* (Rochester, N.Y.)

7767:
★ter Haar, D. Introduction to the physics of many body systems. Interscience Tracts on Physics and Astronomy. No. 5. Interscience Publishers, Inc., New York-London, 1958. viii + 127 pp. Paperbound: $1.95; clothbound: $3.85.

This book containing 125 pages is meant as a short introduction to the modern quantum theory of many-particle systems. On a little less than 40 pages the author sketches the Hartree-Fock approximation, the statistical model of the atom, the nuclear many-body problem (according to Brueckner) and quasi-particles in solids. These topics are collected in part I under the name "Effective field theories". The remaining part of the book is devoted to theories of collective behaviour. This part starts out with a relatively thorough treatment of the general methods in the theories of collective motion, while the applications include sound waves, plasma oscillations, collective behaviour of nuclei and liquid helium. Since many of the mathematical derivations are only sketched, they are sometimes a little difficult to follow. This is of course unavoidable in a book of this size and the natural thing for the reader to do is to go to the original references given in the six pages reference list. The book is excellent as a guide to the current problems in the physics of many-particle systems. *P.-O. Löwdin, J.-L. Calais* (Gainesville, Fla.)

7768:
Brout, R. Variational methods and the nuclear many-body problem. Phys. Rev. (2) **111** (1958), 1324–1333.

"The general form of the energy of the ground state of a

many-fermion system is shown to be exactly of the form proposed by Brueckner and Bethe, without approximation. In a variational treatment, if the trial wave function is picked containing only pair correlations, together with all possible unlinked pairs, it is described by a two-body excitation matrix $(m_1m_2|A|p_1p_2)$. Variation of this matrix in the Ritz-Rayleigh principle yields a set of integral equations of the scattering type for the matrix A. Hole-state energies are given self-consistently in terms of the matrix A, but particle-state energies are Hartree-Fock energies. This may be corrected for by widely enlarging the class of terms admitted into the wave function. If the approximation is then made of omitting a class of terms, defined as cross-linked clusters in $(\psi|H|\psi)$, the particle-state energies are easily renormalized. Variation then leads to an infinite hierarchy of integral equations." (Author's abstract) *P.-O. Löwdin* (Uppsala)

7769:

Chen, Chun-sian; and Chow, Shih-hsun. The basic compensation equation in superconductivity theory when the Coulomb interaction is taken into account. Ž. Èksper. Teoret. Fiz. **36** (1959), 1246–1253 (Russian); translated as Soviet Physics. JETP **9**, 885–889.

The problem of treating simultaneously the Coulomb interaction between electrons and the part of the electron-phonon interaction which leads to superconductivity in a metal is analyzed here. The equations which would have to be solved in a detailed treatment of the problem are derived, and their asymptotic behavior is discussed. It is shown how they lead to Bogoliubov's criterion for the occurrence of superconductivity.

D. J. Thouless (Birmingham)

7770:

Bonč-Bruevič, V. L. On the theory of thermal Green functions. Dokl. Akad. Nauk SSSR **126** (1959), 539–542 (Russian); translated as Soviet Physics. Dokl. **4**, 596–600.

A discussion of temperature-dependent Green functions which are related to (i) one-particle density matrices and more-particle density matrices, and (ii) the quantum mechanical grand partition function. The discussion is restricted to systems of fermions. It is shown that the poles of the Fourier representation of the Green function which may be said to define the effective energies and lifetimes of collective modes (or quasi-particles) are the eigenvalues of a formal "one-particle" problem. The result is applied to a non-degenerate electron gas.

D. ter Haar (Oxford)

7771:

Kogan, Š. M. On thermal quantum Green functions. Dokl. Akad. Nauk SSSR **126** (1959), 546–549 (Russian); translated as Soviet Physics. Dokl. **4**, 604–608.

Discussion intimately related to the one in the paper by Bonch-Bruevich [see preceding review]. The relation between the Fourier transform of the temperature dependent Green function and the quasi-particle energy and its life-time is derived and discussed. The discussion is restricted to the case of an electron gas.

D. ter Haar (Oxford)

7772:

Demkov, Yu. N.; and Ermolaev, A. M. Fock expansion for the wave functions of a system of charged particles. Ž. Èksper. Teoret Fiz. **36** (1959), 896–899 (Russian); translated as Soviet Physics. JETP **9**, 633–635.

It is shown that the wave function of a many-electron system can be expanded as a double series of powers of r and $\ln r$, where r is the square root of the sum of the squares of the distances of the electrons from the nucleus. The equations for the coefficients of the expansion are obtained. This is a generalization of the expansion derived by Fock for the ^1S state of helium.

D. J. Thouless (Birmingham)

7773:

Thouless, David J. Single-particle energies in the many-fermion system. Phys. Rev. (2) **114** (1959), 1383–1390.

The Feynman-Dyson graphical method [F. J. Dyson, Phys. Rev. (2) **75** (1949), 486–502; 1736–1755; MR **10**, 418; **11**, 145] in field theory is first re-interpreted in suitable form for use in non-relativistic many fermion system, similar to Klein and Prange [Phys. Rev. (2) **112** (1958), 994–1007; 1008–1020; MR **21** #5465; #5466]. Using propagators for one-particle (and -hole), the Brueckner approximation for constructing a reaction matrix is explored. The analysis is given using "free"-particle propagator instead of "model"-one-particle propagator, and an approximation is suggested toward rather simple treatment keeping better watch upon the off-energy shell effect on the reaction matrix.

K. Sawada (Princeton, N.J.)

7774:

Vonsovskiĭ, S. V.; and Svirskiĭ, M. S. On the superfluidity of a system of polar Bose excitations. Ž. Èksper. Teoret. Fiz. **36** (1959), 1259–1266 (Russian); translated as Soviet Physics. JETP **9**, 894–898.

The authors consider a model of a monovalent metal in which the energetically most favorable state has all lattice sites occupied by either zero or two valence electrons. Most of the excitations from such a state are boson-like. These bosons interact with one another, and Bogoliubov's theory of interacting bosons can be applied to the system. Under certain conditions, which are described, the system may be superfluid. An interesting difference between this theory of superconductivity and the usual theory is that the attractive interaction between electrons due to the exchange of phonons tends to prevent superconductivity in this theory. The critical velocity is found to depend less strongly on the isotopic mass of the atoms in this theory than in the Bardeen, Cooper, and Schrieffer theory. *D. J. Thouless* (Birmingham)

7775:

Cooper, Leon N. Theory of superconductivity. Amer. J. Phys. **28** (1960), 91–101.

A review paper giving a very clear exposition of the basic considerations underlying the theory of superconductivity in metals developed by Bardeen, Cooper, and Schrieffer [Phys. Rev. (2) **108** (1957), 1175–1204; MR **20** #2196]. *E. L. Hill* (Minneapolis, Minn.)

RELATIVITY

See also 7700.

7776:

★Arzeliès, Henri. La dynamique relativiste et ses applications. Vol. II. Problèmes de mouvement en dynamique du point faiblement accéléré. Avec la collaboration de R. Mendez. Gauthier-Villars, Paris, 1958. xxxiv+451 pp. 6000 francs; $12.46.

A textbook on the mechanics of charged (spinless) particles in static electric and magnetic fields, with relativity corrections taken into account. Linear, resisted motion in a time-varying electric field is also covered; so are collisions and Compton scattering. Motion in an electro-magnetic wave, electron optics and accelerator theory are not included. Radiation effects are ignored.

The methods are essentially pre-relativistic. Relativity is added by inclusion of $(1-\beta^2)^{1/2}$ in the appropriate places, thus emphasizing the differences and complications that arise in relativistic mechanics. The symmetry and elegance of special relativity is lost since the author rejects modern methods of dynamics (Hamilton, Lagrange, Poincaré) and the Minkowski picture as technically useless. A preface, apparently unrelated to the text proper, deals with semi-philosophical problems of "truth and beauty". The author seems to have an axe to grind. His struggling approach to relativity becomes apparent also in a final chapter on the clock paradox.

All chapters are concluded by useful and extensive bibliographies. There are numerous historical annotations and amusing sidelights on the problems in hand. Specific topics (such as, typically, the Stormer orbits) can readily be studied from this book.

O. Buneman (Cambridge, England)

7777:

★Suppes, Patrick. Axioms for relativistic kinematics with or without parity. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 291–307. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

Physics textbooks usually derive the Lorentz transformations from the requirement that they be linear transformations which leave relativistic distances invariant. The author shows that a far weaker requirement is sufficient: Every one-to-one transformation of space-time which preserves the relativistic distance of time-like event intervals is a Lorentz transformation. The invariance of space-like intervals and the linearity can be proved and need not be assumed.

In the final section, the author discusses the question whether it is possible to find "natural" axioms which fix a direction of time. *W. Noll* (Pittsburgh, Pa.)

7778:

Kessler, Paul. Calcul du rayonnement de freinage d'une particule extrême-relativiste par la méthode des processus quasi réels. C. R. Acad. Sci. Paris **249** (1959), 2298–2300.

7779:

Mast, C. B.; and Strathdee, J. On the relativistic interpretation of astronomical observations. Proc. Roy. Soc. London. Ser. A **252** (1959), 476–487.

"The basic ideas of apparent motion are formulated within the framework of general relativity theory. This requires the introduction of a dynamically meaningful frame of reference. A differential equation characterizing motion relative to this frame in a space of arbitrary curvature is formulated and its solutions for nearly flat space-time are obtained. These solutions are compared with the results of classical and special relativistic theory." (Authors' summary) *A. G. Walker* (Seattle, Wash.)

7780:

Callaway, Joseph. Gravitational field of a point charge. Amer. J. Phys. **27** (1959), 469–470.

"A semiclassical derivation of the gravitational potential of a point charge is given. The result agrees with the general theory of relativity except for a factor of $\frac{1}{4}$, which arises as a consequence of the vanishing of the trace of the electromagnetic stress-energy tensor." *Author's summary*

7781:

Bogorodskiĭ, A. F. Relativistic effects in the motion of an artificial earth satellite. Astr. Ž. **36** (1959), 883–889. (Russian. English summary); translated as Soviet Astr. AJ **3**, 857–862 (1960).

"Relativistic perturbations in the motion of an artificial earth satellite are considered. Einstein's field equations, used to derive the law of motion, are employed, while taking into account the rotation of the earth. The equations of motion of a satellite are investigated on the basis of the method of variation of elements. It is shown that secular relativistic perturbations are absent in the semi-major axis, eccentricity and inclination of the orbit to the earth's equator. Secular effects were found in the longitude of the nodes, the displacement of the semimajor axis and the time of perigee passage." *Author's summary*

7782:

Geissler, D. Zum "astronomischen" Bewegungsproblem in der allgemeinen Relativitätstheorie. Z. Naturf. **14a** (1959), 689–696.

The equations of motion for a two-body problem are found according to general relativity theory, under the condition that the gravitational field is weak but the velocities are arbitrarily large. As a parameter of development, the gravitational constant is used. Therefore, in the zeroth approximation, no Newtonian motion is obtained but rectilinear uniform motion is. In the first approximation, the author finds a small deflection from this motion. *L. Infeld* (Warsaw)

7783:

Géhéniau, J. Remarques sur les identités relatives à un lagrangien. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 447–450.

The identities arising from a Lagrangian which depends on a function and its first derivatives are expressed in a

new way. When the Lagrangian is the scalar curvature, one obtains the Møller pseudo-tensor [see C. Møller, Ann. Physics 4 (1958), 347–371; MR 20 #732].

D. W. Sciama (London)

ASTRONOMY

7784:

★**Pecker, J. C.; et Schatzman, E. Astrophysique générale.** Masson et Cie, Paris, 1959. x+756 pp. Broché: 12000 francs; Cart. Toile: 13000 francs.

This is a compendium of most present-day astronomy, the only notable omission being celestial mechanics. The following topics are covered: the theory of statistics, hydrodynamics, turbulence, magneto-hydrodynamics, atomic theory, the interaction of matter and radiation, astronomical instruments, photometry, spectral analysis, the formation of telescopic images, methods of observation with optical telescopes, radio-telescopes, the classification of stellar spectra and their interpretation by the theory of stellar atmospheres, stellar distances, double stars, variable stars, internal constitution of the stars, the structure of the Galaxy, theory of stellar motions, stellar evolution, interstellar matter, external galaxies and cosmology. In addition, the last hundred pages are devoted to the sun, solar-terrestrial relationships, planets and comets.

Mathematical expositions are very brief and there are a certain number of misprints. The book is intended for students in the universities of France, who presumably are able to understand it. In Britain and the United States a great deal of supplementary explanation would be thought necessary before a student could profit from so condensed a work. In English-speaking countries the book would be described as a very valuable and succinct work of reference for professional astronomers.

G. C. McVittie (Urbana, Ill.)

7785:

★**Kurth, Rudolf. Introduction to the mechanics of the solar system.** Pergamon Press, New York-London-Paris-Los Angeles, 1959. ix+177 pp. $6.50.

The principal object of this short volume is to present an outline of the fundamental principles and methods of celestial mechanics, especially to serve young students. The author states in the preface, and repeats on suitable occasions in the book, that he dislikes long formulas; but he also states that the methods derived should be capable of being improved to arbitrary accuracy; for example, by iterative procedures or the inclusion of higher terms by the same method by which they are derived.

The first two chapters, which deal successively with various aspects of elliptic motion, and the final, fourth, chapter on "the planets and the moon as rigid bodies" lend themselves well to this kind of treatment. Especially students who have become familiar with the methods of theoretical physics will feel at home in these chapters.

The important third chapter, "Dynamics of the planetary system" is a brave attempt to give an exposition of some methods of perturbation theory. The presentation is so abbreviated and so incomplete that the chapter falls short of a reasonable goal. An example of a serious misstatement is (on page 86) that if the equations of perturbed planetary motion are written as

$$\ddot{\mathbf{r}} + G\mu \frac{\mathbf{r}}{|\mathbf{r}|^3} = Gm'\left(\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} - \frac{\mathbf{r}'}{|\mathbf{r}'|^3}\right),$$

the right hand members give the differential acceleration $(\Delta\mathbf{r})^{\cdot\cdot}$ that must be added to the Kepler accelerations to obtain the first order approximation. There is nothing in the subsequent pages to indicate to the reader that an additional part,

$$-G\mu\left(\frac{\mathbf{r}_0+\Delta\mathbf{r}}{|\mathbf{r}_0+\Delta\mathbf{r}|^3} - \frac{\mathbf{r}_0}{|\mathbf{r}_0|^3}\right),$$

that arises from the effect of $\Delta\mathbf{r}$ on the Kepler acceleration, contributes an amount of equal magnitude as the part specifically included. Its inclusion is an essential feature of any method of dealing with perturbations in coordinates.

On page 121, after having derived the equations for the variation of the elements, the author remarks that, for commensurability, small divisors appear in the series for the first integral ϕ only in the first power, while in the series for the position vectors these divisors appear squared. He fails to note that the divisors do appear squared when the mean anomaly is integrated by $M = \int n dt + \sigma$, another ignored essential feature.

D. Brouwer (New Haven, Conn.)

7786:

Merman, G. A. Recherches qualitatives dans le problème des trois corps. Byull. Inst. Teoret. Astr. 6, 687–712 (1958). (Russian. French summary)

[Previously reviewed as MR 20 #5684.] The author is concerned with the determination of the final motions (les mouvements finales) in the problem of three bodies when the energy constant is negative. Sufficient conditions are given in order that a motion of the three bodies decomposes into two almost independent motions which are both close to the Keplerian motion. Also estimates are given for the distance and the radial and complete velocities of the body receding to infinity.

The results are obtained by making use of a corollary to a theorem by Birkhoff [*Dynamical systems*, Amer. Math. Soc. Coll. Publ. 9, American Mathematical Society, New York, 1927, p. 276] as given by the author [Byull. Inst. Teoret. Astr. 6 (1955), 232–239; MR 18, 448], and by application of the methods used in the theory of capture of one body by another. *E. Leimanis* (Vancouver, B.C.)

7787:

Merman, G. A. Sur la représentation de la solution générale du problème des trois corps par des séries convergentes. Byull. Inst. Teoret. Astr. 6, 713–732 (1958). (Russian. French summary)

[Previously reviewed as MR 20 #5685.] The author shows the possibility of representing the general solution of the three-body problem by convergent series of an altogether different type than those obtained by Sundman. On the basis of the general results obtained by Sundman he concludes that the Mittag-Leffler star region of the general solution (provided that not all constants of the angular momentum are zero together) contains the whole real w-axis. As a consequence of this fact the general solution can be expanded in series in terms

of polynomials of w, these series being convergent for all real values of w. The sequence of such polynomials $P_n(w)$ can be constructed directly from the corresponding differential equations. The convergence of the polynomial series so obtained follows from general theorems on the existence and prolongability of the solution of the differential equations of the three-body problem (see the preceding review). *E. Leimanis* (Vancouver, B.C.)

7788:

Proskurin, V. F.; and Rumyanceva, L. I. Temporary captures in the problem of three bodies. Byull. Inst. Teoret. Astr. 7, 287–292 (1959). (Russian. English summary)

The authors show the possibility of capture in the following sense. A body, around which a second body moves, is approached by a third body with a hyperbolic velocity. Under the influence of the second body the third body changes the nature of its orbit and moves during a sufficiently large time interval along an elliptic orbit with respect to the first body. A capture of this type is illustrated by an example from the solar system.

E. Leimanis (Vancouver, B.C.)

7789:

Littlewood, J. E. On the equilateral configuration in the restricted problem of three bodies. Proc. London Math. Soc. (3) 9 (1959), 343–372.

This paper deals with the stability problem for the equilateral solutions of Lagrange for the restricted three body problem. This problem is of two degrees of freedom and the energy (in a rotating coordinate system) not positive definite which makes Dirichlet's stability argument inapplicable. The eigenvalues of the linearized equations are denoted by $\pm i\lambda_1$, $\pm i\lambda_2$ which are purely imaginary for sufficiently small μ (which is the ratio of the masses of Jupiter and Sun).

It is a well known unsolved problem to decide whether these equilateral solutions are stable for all times in the sense of Liapounov. Instead the author proposes estimates for long time intervals. Here it is assumed that λ_2/λ_1 is irrational which excludes certain denumerably many values of μ. A typical result is theorem 2: If λ_2/λ_1 is irrational there exist functions $\varphi(\varepsilon)$, $\psi(\varepsilon)$ tending to ∞ faster than any power of ε^{-1} and such that, for $\varepsilon < A_1$ (A_1 is a constant depending on μ only) we have for any t in $0 \leqq t \leqq \varphi(\varepsilon)$ two "asymptotic integrals" of the form

$$\sum_{n=0}^{N-1} \varepsilon^n f_n(x, y, z, w)\Big|_0^t = O(\varepsilon^N) \text{ for } N < \psi(\varepsilon).$$

The rate of increase of $\varphi(\varepsilon)$, $\psi(\varepsilon)$ depends on λ_2/λ_1. Here x, y, z, w are linear combinations of the coordinates x, y and velocities $\dot{x}$, $\dot{y}$ in $\dot{x}^2+\dot{y}^2 \leqq 1$, $x^2+y^2 \leqq 1$. In two other theorems the estimates are improved under the additional assumptions that the partial quotients a_n of the continued fraction of λ_2/λ_1 can be estimated by $a_n = O(n \log^{3/2} n)$. This condition excludes a set of measure zero only. The result can be summarized as follows: If the additional disturbance is of order ε then it remains of order ε over a time interval of order $\exp(A\varepsilon^{-1/2}|\log \varepsilon|^{-3/4})$ where A depends on μ only.

{In this connection the reviewer wants to mention his paper [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa 1955, 87–120; MR 18, 40] where a similar estimate is derived for the periodic solutions of first kind of the restricted three body problem. For that case only finitely many μ values have to be excluded.}

J. Moser (Cambridge, Mass.)

7790:

Kaplan, S. A. On cosmic force-free fields. Astr. Ž. 36 (1959), 800–806. (Russian. English summary); translated as Soviet Astr. AJ 3, 778–783 (1960).

"Cosmic force-free fields rot $H = \alpha H$ with constant α are studied in the most simple configurations: a stratified flat force-free field and a cylindrical helical field. For comparing the theoretical analysis of magnetic fields with observational data it is most convenient to find the correlation between the intensity and polarization of synchrotron radiation. It is shown that for a favorable orientation of the force-free magnetic field (the axis of symmetry is parallel to the plane of the celestial sphere) the polarization of synchrotron radiation does not depend on its intensity and is 100 per cent for the force-free field and 30–40 per cent for cylindrical helical field. If there is a dispersion of orientation, the polarization decreases considerably and gives an inverse correlation with intensity—regions with higher intensity should have smaller polarization.

It is noted that in the force-free field, motion of a shock wave is possible, which increases H and α in the ratio ρ_2/ρ_1 (density discontinuity in a shock wave) and does not distort the force-free character of the field. It is shown that a force-free magnetic field with internal motions of gas is probably unstable. Apparently in cosmic conditions force-free magnetic fields can only exist in rare cases."

From the author's summary

7791:

Agekyan, T. A. The probability of a stellar approach with a given variation of absolute velocity. Astr. Ž. 36 (1959), 41–53. (Russian. English summary)

A star with the mass m and velocity v approaches at distance P another star with the mass m_1, and velocity v_1, the angle between the velocity vectors is α and the squared velocity of the first star is changed from v^2 to $v^2+\Delta v^2$ due to the approach. The gravitational constant is denoted G. At first the author establishes an expression for the probability that the first star during the time dt should approach another star so that P/G lies in the interval $p, p+dp$, m_1/m in the interval $\mu, \mu+d\mu$, v_1/v in the interval $k, k+dk$, the angle between the velocity vectors in the interval $\alpha, \alpha+d\alpha$ and $\Delta v^2/v^2$ in the interval $h, h+dh$. The expression thus obtained is integrated with respect to the parameters p, α, k, and μ, so that the fact that the relative change in the squared velocity $\Delta v^2/v^2$ in time dt takes a value in the interval $h, h+dh$ is established. It is supposed that the velocity distribution for stars of a given mass is Maxwellian. Two simple cases are considered, both involving the supposition of equal masses of the field stars, namely: (1) The star under consideration has the same mass as the field stars; (2) the star under consideration has a small mass compared to that of the field stars. The formulae obtained in these cases involve an incomplete second order moment of a normal distribution.

E. Lyttkens (Uppsala)

GEOPHYSICS

7792:

Businger, J. A. A generalization of the mixing-length concept. J. Meteorol. **16** (1959), 516–523.

Various ways to relate Richardson's number to the vertical wind profile and the concept of mixing length are reviewed under the usual assumptions of steady state and a homogeneous unchanging underlying surface. The mixing length is related to both Eulerian and Lagrangian concepts and to the Reynolds stress. These concepts are then generalized to diabatic conditions, and expressions for the mixing length and the vertical wind profile as a function of a number of parameters and height are obtained. The parameters are k, z_0, T_m, $\partial\bar{\theta}/\partial z$, K, and α. In turn, $\alpha\zeta$ is shown to be a function of α and the Richardson number.

The theory is compared with observations and a reasonable fit is obtained for values of α of 3 and 10. The author states that the present theory is not exact, and that it would be desirable to predict α, which cannot be done at present, but that it is probably not possible to obtain a more precise theory with the mixing length concept until a better knowledge of the ratios of the coefficients of eddy transfer of heat and momentum and of procedures for converting from a Lagrangian to an Eulerian reference system is achieved.

The theory does successfully predict rapid deviations from the neutral profile for only slightly diabatic conditions. *W. J. Pierson, Jr.* (New York, N.Y.)

7793:

Kao, Shih-Kung. Turbulent transfer in the boundary layer of a stratified fluid. J. Meteorol. **16** (1959), 497–503.

Steady flow for a stratified fluid over an infinite uniformly rough plane is treated by adding two parameters related to the buoyancy force and the heat flux to the usual parameter, friction velocity, found in neutral conditions. The gradient of the mean wind, the eddy coefficients of momentum and heat, the mixing lengths for both momentum and heat, the temperature gradient with height and the Richardson number are all then expressed as functions of these three parameters. The results reduce to previously known results for neutral conditions and to a logarithmic plus linear law as a first approximation obtained by others. The velocity profiles converge to those for neutral conditions at higher elevations. For a constant free stream velocity an upward heat flux decreases the friction velocity and a downward heat flux increases it. These results seem to be somewhat more basic and complete than those of the paper reviewed above although the problems in this area are far from completely solved. *W. J. Pierson, Jr.* (New York, N.Y.)

7794:

Lacaze, J. Tourbillons atmosphériques d'axes verticaux (tourbillons de sable). J. Sci. Météorol. **10** (1958), 133–147. (English and Spanish summaries)

7795:

Kuo, H. L. Finite-amplitude three-dimensional harmonic waves on the spherical earth. J. Meteorol. **16** (1959), 524–534.

Under the usual assumptions for large scale atmospheric motions (but retaining a variable Corioles parameter) the author derives a linear equation to describe the propagation of waves of constant angular velocity about the pole. The variations of the amplitude with latitude and vertical height are found, subject to symmetry and boundary conditions imposed. Examples are given of the types of motion that can be constructed by superposition of the fundamental solutions. *O. M. Phillips* (Baltimore, Md.)

7796:

Schwerdtfeger, Werner; and Radok, Uwe. Hodograph analysis as applied to the occurrence of clear-air turbulence. J. Meteorol. **16** (1959), 588–592.

7797:

Cook, A. H. The external gravity field of a rotating spheroid to the order of e^3. Geophys. J. **2** (1959), 199–214.

The author calculates the gravity field due to a rotating spheroid to the third order in e, the eccentricity of the surface. The relation between e, m and J is obtained directly, since the use of McCullagh's theorem involves difficulties of convergence. The gravitational potential is found as a function of spherical polar coordinates and all results are also expressed in terms of the mean radius and gravity. *M. H. Rogers* (Bristol)

7798:

Nickul, K. Über die Darstellung der kleinsten Quadratsumme in einem geodätischen Einpassungsproblem. Soc. Sci. Fenn. Comment. Phys.-Math. **23** (1959), no. 6, 10 pp.

In geodesy it is frequently required to adjust the coordinates of the points of an older net to fit new surveys. The adjustment is customarily done in such a way that the old net is translated, rotated and stretched to obtain the minimum value of the sum of the squares of the differences between the new and the adjusted net.

By vector methods it is shown that "The minimum sum [of the squares of the above-mentioned differences] can be separated into a sum of minimum sums, of which each refers to three points, these points being taken in all combinations". *J. A. O'Keefe* (Chevy Chase, Md.)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

7799:

Stearns, Richard. The voting problem. Amer. Math. Monthly **66** (1959), 761–763.

A group of m voters considers a set A of n alternatives. If $a, b \in A$ we write aPb (a is preferred to b) if a majority of voters prefer a to b and aIb (a and b are indifferent) if exactly half the voters prefer a to b (individual voters are assumed never to be indifferent between alternatives). It is well known that the relation P may exhibit intransitivities, and in fact McGarvey showed [Econometrica **21** (1953), 608–610; MR **15**, 976] that if $m = n(n-1)$ then any pattern of preferences is possible. The present paper shows that

McGarvey's theorem still holds if $m=n+1$ for n odd or $m=n+2$ for m even. These values of m are smallest possible for $n=3, 4, 5$. For large n it is shown that if McGarvey's result is to hold then m must be greater than $.55n/\log n$. *D. Gale* (Providence, R.I.)

7800:

★**Wold, Herman O. A. Ends and means in econometric model building.** Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 355–434. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

A lengthy discursive review of the relative merits of causal chains and interdependent systems as econometric models. *J. Durbin* (Chapel Hill, N.C.)

7801:

Newell, G. F. A theory of platoon formation in tunnel traffic. Operations Res. **7** (1959), 589–598.

"A crude mathematical model is constructed to represent the flow of traffic in tunnels or other finite sections of highway on which passing is impossible. The main ingredients of the model are: (1) each car has a 'desired speed' which is considered to be a random variable, (2) the actual speed of any car is the smaller of its own desired speed and the actual speed of the car ahead of it, (3) if the latter speed is the smaller, the trailing car maintains some fixed spacing Δ behind its predecessor. The consequences of the model are: (1) the capacity of a tunnel is a monotone decreasing function of its length, (2) the velocity of any car is an increasing function of its distance from the entrance and is equal to the smallest of its own desired speed and those of all other cars in the tunnel ahead of it, (3) the spacing between cars is equal to Δ for most cars but when the spacing is larger than Δ, it is usually much larger and increases with the distance from the entrance." (From the author's summary.) The qualitative comparison of results with data from the Lincoln Tunnel in New York is very encouraging, but in two other tunnels considered, the flow seems to be dominated by other effects.

G. B. Whitham (Cambridge, Mass.)

7802:

Morse, Philip M. Solutions of a class of discrete-time inventory problems. Operations Res. **7** (1959), 67–78.

The usual discrete-time inventory model is studied with demand a discrete variable independent from period to period. However, ordering takes place only once every T periods. The stationary solution is found for the policy of ordering so as to seek a fixed level of inventories. Some variations are introduced, particularly with respect to lags in replenishment.

Finally, asymptotic formulas are found, for large mean demands, for the distribution of demand in a period starting from assumptions of independent successive arrivals.

K. J. Arrow (Stanford, Calif.)

7803:

Scarf, Herbert. Bayes solutions of the statistical inventory problem. Ann. Math. Statist. **30** (1959), 490–508.

Consider an inventory problem with no backlogging, linear ordering, holding and penalty costs, a constant rate of discount, and an infinite horizon. Demands in different periods are assumed to be independent random variables with the same distribution. If the distribution of demand were known, there would be a single number $\bar{x}$ such that the optimal ordering policy would consist of ordering $\bar{x}-x$ when the inventory was $x<\bar{x}$ and not ordering otherwise.

The author considers the problem where the density of demand ξ is not known a priori, but belongs to the exponential family, $\phi(\xi, \omega)=\beta(\omega)r(\xi)\exp(-\omega\xi)$, where ω is a parameter with an a priori distribution. As time goes on, more information is obtained about ω; at any time n, it is summed up in the single statistic, $s=\sum_{i=1}^{n-1}\xi_i/(n-1)$. It is shown that the optimal rule is defined by an infinite sequence of functions, $\bar{x}_n(s)$ such that the amount ordered is $\max[\bar{x}_n(s)-x, 0]$ at time n. The functions $\bar{x}_n(s)$ are monotone increasing. Further, they converge, as n approaches infinity, to $\bar{x}(s)$, which is the critical value defining the optimal policy if the parameter were known to have the value s. The second term of an asymptotic expansion is also obtained.

K. J. Arrow (Stanford, Calif.)

7804:

Connell, Ian G. A generalization of Wythoff's game. Canad. Math. Bull. **2** (1959), 181–190.

The author defines a K-Wythoff game, a modification of the game of Nim, by the following rules: (i) there are two players who play alternately; (ii) initially there are two piles of matches, an arbitrary number in each pile; (iii) a player may take an equal number of matches from both piles or a multiple of K matches from one pile, but he must take at least one match; (iv) the player who cannot make a legitimate move loses the game. The author obtains the set of all winning pairs for such games. This set has the following properties: (a) an arbitrary pair is in the set or can be reduced to a pair in the set in just one move; (b) a move on a pair in the set produces a pair not in the set; (c) all pairs on which no legitimate move can be made are in the set.

M. Dresher (Pacific Palisades, Calif.)

7805:

Connell, Ian G. Some properties of Beatty sequences. I. Canad. Math. Bull. **2** (1959), 190–197.

The pair of sequences $U_n=[n(1+\alpha)/\alpha]$, $V_n=[n(1+\alpha)]$, $n=1, 2, 3, \cdots$, (where square brackets denote the integral part function) contain all the positive integers without repetition or omission if and only if $\alpha>0$ and α is irrational. The author calls the pair Beatty sequences of argument α. Beatty sequences of particular arguments α are connected with Wythoff's game, a generalization of the game of Nim [see preceding review]. The main object of the paper is to determine all $\alpha>0$ for which the Beatty sequences of argument α satisfy $U_{V_n}=U_n+V_n$ for every positive integer n. *M. Dresher* (Pacific Palisades, Calif.)

7806:

★**Hannan, James. Approximation to Bayes risk in repeated play.** Contributions to the theory of games, vol. 3, pp. 97–139. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N.J., 1957. $5.00.

Consider N successive plays of an $m\times n$ matrix game in which player I's "motivation" is unspecified. As a

criterion for a sequence strategy for player II, consider the expected inutility of the sequence strategy over the N plays minus N times the inutility of the single game Bayes strategy against I's empirical distribution of choices over the entire N plays. A sequence strategy that minimizes the maximum of this criterion is characterized recursively, and it is shown that except when II has a dominant column the min-max value is bounded from below by a non-zero multiple of $N^{1/2}$. This strategy is developed in detail for a generalization of matching pennies. For the sequence strategy in which at the $(k+1)$st stage II plays Bayes against I's cumulative past choices plus

$$(3n^2/2m)^{1/2}k^{1/2}z,$$

where z is chosen at random from the unit cube, then an upper bound is $(3n^2m/2)^{1/2}|B|N^{1/2}$, where $|B|$ denotes the maximum difference within II's rows. Upper bounds are also obtained for Bayes strategies against cumulative past choices when the game is non-finite provided the Bayes response satisfies a Lipschitz condition.

R. D. Luce (Philadelphia, Pa.)

BIOLOGY AND SOCIOLOGY

See also 7567, 7608, 7610.

7807:

Cane, Violet R. Behaviour sequences as semi-Markov chains. J. Roy. Statist. Soc. Ser. B **21** (1959), 36–58.

A discussion of the use of simple probability models to summarize data from certain experiments on birds, fish and insects, in which long records of steady state behaviour are obtained. Particular emphasis is placed on the use of semi-Markov chains. The way in which these models can be used to test ethological theories is also discussed.

J. L. Snell (Hanover, N.H.)

7808:

Polonsky, J. Essai d'interprétation du fonctionnement des cellules vivantes dans le cadre de la cybernétique quantique. Cybernetica **2** (1959), 75–97, 162–194.

7809:

Gibson, W. A. Three multivariate models: factor analysis, latent structure analysis, and latent profile analysis. Psychometrika **24** (1959), 229–252.

The author presents a method, called latent profile analysis, for the study of interrelations among quantitative measures, such as test scores. The method is a generalization of Lazarsfeld's latent structure analysis, and is intended to provide an alternative to factor analysis. The underlying model is the following. The population under study is assumed to be composed of latent groups, of unknown sizes $n_1, \cdots, n_q$. Within each group, the different test scores $x_1, \cdots, x_s$ are assumed to be independent. Let the group means of these scores be X_{ij} $(i=1, \cdots, q; j=1, \cdots, s)$. It then follows that the product sums n, m_j, m_{jk}, $\cdots$ of 0, 1, $\cdots$ scores, over the whole population, are of form

$$n = n_1+n_2+\cdots+n_q,$$
$$m_j = n_1X_{1j}+\cdots+n_qX_{qj} \quad (j = 1, \cdots, s),$$
$$m_{jk} = n_1X_{1j}X_{1k}+\cdots+n_qX_{qj}X_{qk} \quad (j, k = 1, \cdots, s),$$

etc. Under appropriate assumptions on q and s, the unknown quantities on the right hand side are calculable from the observable quantities on the left hand side.

G. Elfving (Helsinki)

INFORMATION AND COMMUNICATION THEORY

See also 7608.

7810:

Hartmanis, Juris. The application of some basic inequalities for entropy. Information and Control **2** (1959), 199–213.

"For a stationary ergodic source the usual entropy rate $\lim H_n/n$ describes only some aspects of the source. For storage or processing applications the related quantities H_n, ΔH_n, and $\Delta^2 H_n$ play an important role and are closely connected with the 'complexity' of the source." (From the author's summary) *S. P. Lloyd* (Murray Hill, N.J.)

7811:

★Rothstein, Jerome. Communication, organization, and science. With a foreword by C. A. Muses. The Falcon's Wing Press, Indian Hills, Colo., 1958. xcvi+110 pp. $3.50.

Chapter headings are: Information and entropy; Measurement and communication; Information and thermodynamics; Information, logic, and physics; Language, science, and art; Information, organization, and systems; Theory as organization of observation and a system for prediction; Skill as organization; Miscellaneous applications of the organization concept; Science, value, and games; On the behavior of organized mechanisms; On the possibility of constructing a universal language; Toward a unified world outlook.

S. P. Lloyd (Murray Hill, N.J.)

7812:

Hagelbarger, D. W. Recurrent codes: easily mechanized, burst-correcting, binary codes. Bell System Tech. J. **38** (1959), 969–984.

Errors in the transmission of an indefinitely long sequence of binary digits are assumed to occur in "bursts", where a burst of length l is a subsequence of l digits such that the first, last, and possibly some intermediate digits are incorrectly transmitted. A class of coding schemes is discussed which permits the decoder to correct bursts of length l or less, provided the number of correctly transmited digits occurring between bursts is never less than a certain minimum. This minimum, called the "guard space", depends upon the method of coding.

Coding is achieved by introducing check digits into the sequence at regular intervals before transmission. These check bits are formed as modulo 2 sums of message digits of the sequence in certain specific relative positions.

A coding and decoding scheme is described for arbitrary l and frequency f of check digits, and mechanisms for coding and decoding are given schematically. Special attention is paid to economy in mechanization and to synchronization problems. *D. E. Muller* (Urbana, Ill.)

7813:

Stam, A. J. Some inequalities satisfied by the quantities of information of Fisher and Shannon. Information and Control **2** (1959), 101–112.

7814:

Cox, D. R.; and Brandwood, L. On a discriminatory problem connected with the works of Plato. J. Roy. Statist. Soc. Ser. B **21** (1959), 195–200.

The authors analyze statistically the distribution of sentence endings in seven dialogues of Plato to obtain a reasonable chronological ordering of them. The final ordering is *Republic*, *Timaeus*, *Sophist*, *Critias*, *Politicus*, *Philebus*, and *Laws*, with the position of *Critias* most in doubt. Its position could be from some time before *Timaeus* to some time before *Politicus*.

E. J. Cogan (Bronxville, N.Y.)

7815:

Chomsky, Noam. A note on phrase structure grammars. Information and Control **2** (1959), 393–395.

In the author's paper in Information and Control **2** (1959), 137–167 [MR **21** #4107] a precise characterization of nonfinite state languages that can be generated by type 2 (here called 'context-free') grammars is obtained. As a step toward this, the following theorem is proved: If G is a context-free grammar generating L_G and every nonterminal derivable string is of the form xA or every nonterminal derivable string is of the form Ax, then there is a finite Markov source that generates L_G. The present paper is devoted to a new, shorter proof of this theorem, using notation developed in the paper cited.

E. J. Cogan (Bronxville, N.Y.)

7816:

★Booth, Andrew D.; Brandwood, L.; and Cleave, J. P. Mechanical resolution of linguistic problems. Academic Press Inc., New York; Butterworths Scientific Publications, London, 1958. vii+306 pp. $9.80.

This volume contains a comprehensive account of recent work on mechanical translation from one language to another by use of high speed calculating machines. The methods of translation used at Birkbeck College are presented together with an account of their tests on an actual computing machine.

The first chapter contains a brief but intelligible description of the nature of calculating and data-processing machines. The second deals with the analysis of statistical problems in linguistics, word counting and analysis of sentence structure. The third chapter discusses the problems of stylistic analysis, syntax and the use of dictionaries. Next, a programming technique for mechanical translation is outlined, followed by discussion of transcription of the Braille system. Chapters eight and nine discuss specifically the systems used in Birkbeck College for translation from French to English, and in great detail, a treatment based on the work by Oswald and Fletcher on the German syntax.

Chapter ten gives an account of the work of the Academy of Sciences in the USSR in Moscow on the mechanical translation of English into Russian. The last chapter outlines features of a machine to be built specifically for translation.

The book constitutes an excellent introduction into this growing field, posing problems in information theory, mathematical logic and statistical studies in linguistics.

S. M. Ulam (Los Alamos, N.M.)

SERVOMECHANISMS AND CONTROL

See also 7608, 7808.

7817:

Letov, A. M. The problem of quality for non-linear self-regulating systems with quadratic metric. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Energet. Avtomat. **1959**, no. 3, 25–31. (Russian)

Consider a system characterized by the equations

$$\dot{x}_k = \sum_a b_{ka}x_a + f_k(x_1, \cdots, x_n, t) \quad (k = 1, \cdots, n),$$

where the b_{ka} are constants and the f_k are well-behaved functions with the properties: (a) $f_k(0, \cdots, 0, t) \equiv 0$ for t in $[0, T]$ and $k = 1, \cdots, n$; (b) there exist positive numbers L_k such that throughout the domain of definition of the f_k, $|f_k(x_1, \cdots, x_n, t)| < L_k R$, where $R^2 = x_1^2 + \cdots + x_n^2$. The quality of performance is measured in terms of the rapidity with which the system returns to equilibrium and the extent of overshoot in phase space. The problem is to determine the values of adjustable parameters $p_1, \cdots, p_m$ in the b_{ka} and f_k which yield optimal performance. The author's results are too complicated for reproduction; essentially, they involve maximizing the smallest eigenvalue of a quadratic form with known coefficients on a quadratic surface in phase space.

L. A. Zadeh (Berkeley, Calif.)

7818:

Krasovskiĭ, A. A. Dynamics of continuous systems of extremal regulation, based on the gradient method. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Energet. Avtomat. **1959**, no. 3, 43–49. (Russian)

A brief analysis of the application of gradient methods to the maximization and minimization of n-ary functions $F(x_1, x_2, \cdots, x_n)$. The partial derivatives $\partial F/\partial x_i$ are measured by means of small sinusoidal signals superimposed on constant values of the x_i, and F is approximated by a quadratic form. This leads to a linear feedback system realization for the extremizing process. The question of stability of such realizations is considered. However, the effects of noise, which are of importance in problems of this nature, are not analyzed.

L. A. Zadeh (Berkeley, Calif.)

7819:

Smolickiĭ, H. L.; and Čukreev, P. A. On the optimal reservation of apparatus. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Energet. Avtomat. **1959**, no. 4, 79–85. (Russian)

The element-by-element reservation of apparatus, when its weight, gabarit, and cost are subject to limiting conditions, is important in the construction and formulation of automated systems. This paper presents a generalized method for the solution of this problem.

A system is assumed to consist of n elements. Their

respective weights are w_i, respective probabilities of acceptance are p_i and probabilities of rejection are q_i. The optimum reservation for a predetermined probability-value of the proper work of a system resolves in the consideration of $p_{\text{reserved}} = \prod_{i=1}^{n}(1-q_i^{k_i}) \geq p_{\text{required}}$ and $w_{\text{reserved}} = \sum_{i=1} k_i w_i = \min$.

The solutions are given in terms of extremal values (but due to the nature of the problem, in integral values only). The criteria are:

$$(1) \qquad \sum_{i=1}^{n} (k_i - k_i^0)w_i \geq 0, \quad \sum_{i=1}^{n} (k_i - k_i^0)w_i = \min.$$

$$(2) \qquad \prod_{i=1}^{n} (1-q_i k_i) \geq p_{\text{required}}.$$

The resulting equation is

$$p_{\text{required}} = y^n / \prod_{i=1}^{n} (y + a_i).$$

A. Bakst (Flushing, N.Y.)

7820:

Constantinescu, Paul. Sur le décroissement du nombre des contacts en utilisant circuites en pont. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 11, 45–69. (Romanian. French and Russian summaries)

Suppose, between two nodes of a switching circuit, one has two paths where the conductivity (the passing or non-passing of current) of the second path is some Boolean function QR, where Q and R are functions of the direct conductivities of the contacts, the conductivities between adjacent nodes. If an extra node is inserted into this second path in such a way as to break it up into two parts in series with conductivities Q and R, the author shows that the total conductivities between any pair of original nodes are conserved. Suppose that a circuit between two nodes consists of a series of $r+1$ parallel pairs of contact schemes with the parallel branches having conductivities S_i and T_i $(i=1, 2, \cdots, r+1)$ and also in this series one parallel scheme where one branch has conductivity R, the other, $S_1S_2\cdots S_{r+1}$. The second theorem is that if one removes the branch with this latter product conductivity from the circuit, then the total conductivity between any two remaining nodes is conserved. Finally, consider a circuit which consists of $n+1$ paths between two nodes 1 and 2. Suppose that each of the last n paths breaks up into an upper (connected with 1) and a lower part. Suppose, further, that each such upper part decomposes into parallel branches, the conductivity of one of these branches always matching the conductivity of the corresponding lower portion of the path to within a fixed factor S; and let the first path have total conductivity T. Replace the first path by a parallel pair with conductivities T and S and then remove each branch of the upper portions of the last n paths with the product conductivities. Then, excluding the node 2 from the discussion, the total conductivity between each pair of nodes is unchanged. Reference is made to M. A. Gavrilov [*Teoriya releĭno-kontaktnyh shem*, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950; MR **12**, 225]. A number of examples are given. There are errors in the printing of subscripts, and the two summaries are somewhat less than helpful. *F. Haimo* (St. Louis, Mo.)

7821:

Constantinescu, Paul. Sur des schémas obtenus en employant les congruences des entiers dans la théorie des mécanismes automatiques. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 12, 23–32. (Romanian. French and Russian summaries)

Let the total conductivity across a two-terminal relay-contact scheme be given by $x_1 \oplus x_2 \oplus \cdots \oplus x_n$, where the latter is $x_1 \oplus (x_2 \oplus \cdots \oplus x_n)$ and where $x_1 \oplus x_2$ is the Boolean function of the Boolean variables x_1, x_2 (values chosen from the two-element Boolean algebra) given by $x_1x_2' + x_1'x_2$. Here, x' is the complement of x in the Boolean algebra. The author shows that a predetermined total conductivity $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ from terminal to terminal can always be realized by a scheme with $4(n-1)$ contacts and with $2n$ nodes (including the two terminals). See C. Cardot [Ann. Télécommun. **7** (1952), 75–84; MR **16**, 314] and the preceding review. On p. 24, there are annoying minor printing errors.

F. Haimo (St. Louis, Mo.)

7822:

Constantinescu, Paul. Analyse des schémas Π et H à éléments soupape. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 165–172. (Romanian. Russian and French summaries)

The author shows how to determine the existence of a closed path between two nodes of a certain switching circuit, where the open or closed status of the various contacts and the orientations of the rectifiers are known, by the method of minors as described in Aranovič [Avtomat. i Telemeh. **10** (1949), 437–451; MR **11**, 574] and in Lunc [Dokl. Akad. Nauk SSSR **70** (1950), 421–423; MR **11**, 574]. The second set of problems is to determine the status of the contacts (the orientation of the rectifiers being known) which allows the current to flow (a) in either direction between two given nodes, (b) in one given direction or (c) in neither direction. Four H relay-contact schemes and one Π scheme are so analyzed by considering the matrices of direct conductivities with entries from the Boolean algebra of two elements.

F. Haimo (St. Louis, Mo.)

7823:

Vasil'ev, Yu. L. Minimal contact schemes for Boolean functions of four variables. Dokl. Akad. Nauk SSSR **127** (1959), 242–245. (Russian)

The author obtains the number $L(f)$ of contacts in a minimal contact scheme realizing a function f of each of the 402 types of functions of four arguments. The general technique throughout consists in finding a lower bound for $L(f)$ and exhibiting a scheme which realizes f and which uses a number of contacts equal to the lower bound. Criteria for obtaining lower bounds are obtained in essentially two ways. One is associated with an analysis of the "defects" in the structure of a given scheme, and the other with an n-dimensional hypercube related to the function to be realized. Functions of forty types were treated individually since the above techniques yielded no results for them. The maximum number of contacts needed is 13, and that only for two types. Tables are provided which summarize the results for all 402 types of four argument functions. *E. J. Cogan* (Bronxville, N.Y.)

7824:

Talancev, A. D. Analysis and synthesis of certain electric circuits by means of special logical operators. Avtomat. i Telemeh. **20** (1959), 898–907. (Russian. English summary)

For the analysis and synthesis of sequential circuits whose transitions take place only at multiples of a clock time interval, the author introduces two special operators d and D, which indicate special kinds of differentiation with respect to time. These operators d and D are applicable both to Boolean functions $X(t)$ of time which remain constant from one clock pulse time to the next, and to Boolean functions $Y(t)$ which take on the value 1 only for brief pulses at clock pulse times. Unlike the customary Boolean difference, these operators have the property that any Boolean function Z of time satisfies $d^2Z=0$ and $D^2Z=0$. Also d has the property that not all Boolean functions $Y(t)$ have corresponding "integrated" functions F such that $Y=dF$. This paper gives a procedure for solving Boolean equations to perform the integration whenever this is possible, and uses this procedure to design an example of a circuit.

E. F. Moore (Murray Hill, N.J.)

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